Regular second order perturbations of binary black holes: The extreme mass ratio regime

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Abstract. In order to derive the precise gravitational waveforms for extreme mass ratio inspirals (EMRI), we develop a formulation for the second order metric perturbations produced by a point particle moving in the Schwarzschild spacetime. The second order waveforms satisfy a wave equation with an effective source build up from products of the first order perturbations and its derivatives. We have explicitly regularized this source at the horizon and at spatial infinity. We show that the effective source does not contain squares of the Dirac's delta and that perturbations are regular at the particle location. We introduce an asymptotically flat gauge for the radiation fields and the $\ell = 0$ mode to compute explicitly the (leading) second order $\ell = 2$ waveforms in the headon collision case. This case represents the first completion of the radiation reaction program self-consistently.

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1. Introduction

In the past twenty years we have witnessed steady increase in the interest in gravitational waves from astrophysical sources. Specially driven by the design and construction of laser interferometric detectors, both ground and space based. Alongside with this experimental developments theoretical progress has also been steady. We are now in conditions to predict the gravitational radiation from the astrophysical scenarios expected to produce the strongest signals, i.e. the merging binary black holes.

The two main scenarios involving black holes are, first, galactic binaries with black holes having comparable masses (a few solar masses). They are, for instance, the product of a supernova explosion plus a subsequent accretion. The second scenario involves a supermassive black holes (with several million solar masses) residing in the center of an active galaxy. They attract stars in the inner nuclei towards unstable orbits with a subsequent plunge generating observable gravitational radiation. This scenario clearly involves extreme mass ratio collisions. Let us also mention that a less common event, but most energetic, is the close collision of galaxies and consequently of supermassive black holes in their respective cores.

From the theoretical point of view one advantage of dealing with binary black holes is that one can treat the problem of generation of radiation in terms of only its gravitational field, ignoring the (small) effects of matter around the binary system. A second important feature is that the equations of General Relativity scale with the total mass of the system. In this way, one can choose the dimensions of the systems a posteriori, i.e. after solving for the scale free problem (See [1] for the case of three black holes.). It is then convenient to characterize the binary black hole systems in terms of the mass ratio of its components (besides the individual spins, orbital parameters and spatial orientation with respect to the observer).

In binary black hole systems most of the generation of gravitational radiation take place during the final few orbits before the merger. In the case of comparable masses, this stage involves highly nonlinear interactions among black holes and can only be described by directly solving numerically the full General Relativity field equations. Until recently this represented an insurmountable task that hold the field for nearly thirty years, but during the year 2005 two successful approaches [2, 3, 4] have lead to stable codes that allow to simulate binary black holes in supercomputers. This breakthrough in Numerical Relativity lead to numerous studies during the last year, including the last few quasi-circular orbits [5, 6], the effect of ellipticity [7], and spin-orbit coupling leading to a change in the merger time [8], corotation [9] and spin-flip and precession [10]. Finally, unequal mass black holes have been studied in [11, 12, 13, 14] reaching a minimum mass ratio of nearly 1:4. But simulations with mass ratios up to 1:10 are currently underway. Generic binary simulations, i.e. unequal masses and spins, have first been reported in [15], this simulation lead to the shocking discovery that merging spinning black holes can acquire recoil velocities up tp 4000 km/s [16]. Even multi-black hole spacetimes are now possible to evolve numerically [17, 18].

In the small mass ratio regime the smaller hole orbiting the larger is considered as a perturbation. In this approach the smaller hole is described by a Dirac's delta particle and the spacetime is no longer empty but has a non-vanishing energy-momentum tensor at the location of the particle. The simplicity of this approach is appealing, but the problem notably complicates when self-force effects are taken into account to compute the correction to the background geodesic motion. A consistent approach to this problem has been first laid down ten years ago by Mino, Sasaki and Tanaka [19] and later confirmed by Quinn and Wald [20] by providing a regularization procedure. (See [21, 22] for a detailed review and [23] for a practical regularization method.) Self-force corrections can be considered second order effects on the mass ratio of the holes, the natural perturbation parameter for the system. To consistently compute the gravitational waveform, radiation energy, angular and linear momentum radiated to infinity (and onto the horizon) we need to proceed to solve the second order perturbations problem for the gravitational field. The formalism to study second (and higher) order perturbations of rotating black holes with sources was extensively discussed in [24] based on the Newman-Penrose approach of curvature perturbations.

The first explicit computation of the gravitational self-force was done for the headon collision of two black holes in Ref. [25, 26]. This allow us to complete here, for the first time, the second order program applied to the headon collision of black holes. Note that this step is also crucial in order to close the gap between the full numerical simulation with comparable masses and the perturbative approach for extreme mass ratios.

Here, the second order calculation is required to derive precise gravitational waveforms which are used as templates for gravitational wave data analysis. In general, this second order computation has to be done by numerical integration. It is hence important to derive a well-behaved second order effective source, and we will focus on this problem in this paper. The second order analysis was pioneered by Tomita [27, 28], and vacuum perturbations in the Schwarzschild background was studied by Gleiser *et al.* [29, 30, 31, 32]. There are also studies on second order quasi-normal modes [33, 34], and the second order analysis was also extended to cosmology [35, 36, 37, 38, 39].

This paper is organized as follows. We consider second order metric perturbations and the equations they satisfy, i.e., the perturbed Hilbert-Einstein equations of General Relativity in section 2. In section 3, we discuss the first order metric perturbations in the case of a particle falling radially into a Schwarzschild black hole by using the Regge-Wheeler-Zerilli formalism [40, 41]. This formalism in the time domain is summarized in Appendix A. To calculate the second order source, the first order $\ell = 0$ mode is given in an appropriate gauge in Appendix C. In section 4, we derive the regularized second order effective source where the singular behaviors are removed analytically. In Sec. 5, we summarize this paper and discuss some remaining open problems. Details of the calculations are given in the appendices. Throughout this paper, we use units in which $c = G = 1$.

2. Second order metric perturbations

We consider second order metric perturbations on black hole backgrounds,

$$\tilde{g}_{\mu\nu} = g_{\mu\nu} + h^{(1)}_{\mu\nu} + h^{(2)}_{\mu\nu} \,, \tag{1}$$

with expansion parameter μ/M corresponding to the mass ratio of the holes. Here, $g_{\mu\nu}$ is the background metric, and superscripts (i) $(i = 1,\, 2)$ denote the perturbative order, i.e., $h^{(1)}_{\mu\nu}$ and $h^{(2)}_{\mu\nu}$ are called the first and second order metric perturbations, respectively. In the perturbative calculation, we raise and lower all tensor indices with the background metric. The Hilbert-Einstein tensor up to the second perturbative order is given by

$$G_{\mu\nu}[\tilde{g}_{\mu\nu}] = G^{(1)}_{\mu\nu}[h^{(1)}] + G^{(1)}_{\mu\nu}[h^{(2)}] + G^{(2)}_{\mu\nu}[h^{(1)}, h^{(1)}] \,, \tag{2}$$

where we have omitted the spacetime indices μ and ν of the metric perturbations, $h^{(1)}_{\mu\nu}$ and $h^{(2)}_{\mu\nu}$, and ignored $\mathrm{O}((\mu/M)^3)$ and higher order terms. $G^{(1)}_{\mu\nu}$ is the well known linearized Hilbert-Einstein tensor,

$$G^{(1)}_{\mu\nu}[H] = -\frac{1}{2} H_{\mu\nu;\alpha}{}^{;\alpha} + H_{\alpha(\mu;\nu)}{}^{;\alpha} - R_{\alpha\mu\beta\nu} H^{\alpha\beta} - \frac{1}{2} H_{\alpha}{}^{\alpha}{}_{;\mu\nu} - \frac{1}{2} g_{\mu\nu} (H_{\lambda\alpha}{}^{;\alpha\lambda} - H_{\alpha}{}^{\alpha}{}_{;\lambda}{}^{;\lambda}) \,, \tag{3}$$

Here, $H_{\mu\nu}$ denotes $h^{(1)}_{\mu\nu}$ or $h^{(2)}_{\mu\nu}$, and semicolon ";" in the index denotes the covariant derivative with respect to the background metric. $G^{(2)}_{\mu\nu}$ consists of quadratic terms in the first order perturbations,

$$G^{(2)}_{\mu\nu}[h^{(1)}, h^{(1)}] = R^{(2)}_{\mu\nu}[h^{(1)}, h^{(1)}] - \frac{1}{2} g_{\mu\nu} R^{(2)\alpha}_{\alpha}[h^{(1)}, h^{(1)}] \,; \tag{4}$$

$$\begin{aligned} R^{(2)}_{\mu\nu}[h^{(1)}, h^{(1)}] &= \frac{1}{4} h^{(1)}_{\alpha\beta;\mu} h^{(1)\alpha\beta}{}_{;\nu} + \frac{1}{2} h^{(1)\alpha\beta} (h^{(1)}_{\alpha\beta;\mu\nu} + h^{(1)}_{\mu\nu;\alpha\beta} - 2 h^{(1)}_{\alpha(\mu;\nu)\beta}) \\ &\quad - \frac{1}{2} (h^{(1)\alpha\beta}{}_{;\beta} - \frac{1}{2} h^{(1)\beta;\alpha}_{\beta}) (2 h^{(1)}_{\alpha(\mu;\nu)} - h^{(1)}_{\mu\nu;\alpha}) + \frac{1}{2} h^{(1)}_{\mu\alpha;\beta} h^{(1)\alpha;\beta}_{\nu} - \frac{1}{2} h^{(1)}_{\mu\alpha;\beta} h^{(1)\beta;\alpha}_{\nu} \,. \end{aligned}$$

On the other hand, the stress-energy tensor includes three parts;

$$T_{\mu\nu} = T^{(1)}_{\mu\nu} + T^{(2,SF)}_{\mu\nu} + T^{(2,h)}_{\mu\nu} \,, \tag{5}$$

The first order stress-energy tensor $T^{(1)}_{\mu\nu}$ which is the one of a point particle moving along a background geodesic, is given by

$$T^{(1)\mu\nu} = \mu \int_{-\infty}^{+\infty} \delta^{(4)}(x - z(\tau)) \frac{dz^\mu}{d\tau} \frac{dz^\nu}{d\tau} d\tau \,, \tag{6}$$

where

$$z^\mu = \{T(\tau), R(\tau), \Theta(\tau), \Phi(\tau)\} \,, \tag{7}$$

for the particle's orbit. $T^{(2,SF)}_{\mu\nu}$ denotes the deviation from the geodesic by the self-force as derived by the MiSaTaQuWa formalism [19, 20]. We do not treat this stress-energy tensor explicitly in this paper. And $T^{(2,h)}_{\mu\nu}$, which is purely affected by the first order metric perturbations, is written as

$$T^{(2,h)}_{\mu\nu} = -\frac{1}{2} \mu \int_{-\infty}^{+\infty} h^{(1)\alpha}_{\alpha} \, \delta^{(4)}(x - z(\tau)) \frac{dz^\mu}{d\tau} \frac{dz^\nu}{d\tau} d\tau \,, \tag{8}$$

where we have used the determinant

$$\tilde{g} = g(1 + h_{\alpha}^{(1)\alpha})\,, \tag{9}$$

up to the first perturbative order.

With the above expansion of the Hilbert-Einstein and stress-energy tensors, we may solve the following equation for the first perturbative order,

$$G_{\mu\nu}^{(1)}[h^{(1)}] = 8\,\pi\, T_{\mu\nu}^{(1)}\,. \tag{10}$$

And for the second perturbative order, we have the following equation.

$$G_{\mu\nu}^{(1)}[h^{(2)}] = 8\,\pi\,\left(T_{\mu\nu}^{(2,SF)} + T_{\mu\nu}^{(2,h)}\right) - G_{\mu\nu}^{(2)}[h^{(1)}, h^{(1)}]\,. \tag{11}$$

Once the first order metric perturbations $h^{(1)}$ (and the self-force) are obtained, we may solve (11) with a second order source that can be considered as an effective stress-energy tensor. Systematically expanding the Hilbert-Einstein equations, one can obtain the perturbative equations order by order [42, 24, 43, 44, 45].

3. First order perturbations in the Regge-Wheeler gauge

In this paper, we consider the Schwarzschild background,

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)\,. \tag{12}$$

in Boyer-Lindquist coordinates.

Before considering the second perturbative order, it is necessary to discuss the first order metric perturbations, i.e., the first order Hilbert-Einstein equation given in (10). The Regge-Wheeler-Zerilli formalism [40, 41] is used here. The basic formalism has been given in Zerilli's paper [41], and it has been summarized in the time domain in [46, 47]. In Appendix A, we establish our notation and summarize the Regge-Wheeler-Zerilli formalism in the time domain.

The treatment of the first perturbative order is as follows. First, we expand $h_{\mu\nu}^{(1)}$ and $T_{\mu\nu}^{(1)}$ in ten tensor harmonics components, given in (A.1) and (A.2). We then obtain the linearized field equations for each harmonic mode. Here, for example, for the even parity modes which have the even parity behavior, $(-1)^{\ell}$ under the transformation $(\theta, \phi) \to (\pi - \theta, \phi + \pi)$, we may consider the Zerilli equation in (A.8). Finally, imposing the Regge-Wheeler (RW) gauge conditions:

$$h_{0\,\ell m}^{(e)(1)RW} = h_{1\,\ell m}^{(e)(1)RW} = G_{\ell m}^{(1)RW} = 0\,, \tag{13}$$

where the suffix RW stands for the RW gauge, we obtain the first order metric perturbations as in (A.11).

3.1. Geodesic motion and the first order stress-energy tensor

We consider a particle falling radially into a Schwarzschild black hole as the first order source. Assuming $\Theta(\tau) = \Phi(\tau) = 0$, the equation of motion of the particle is given as

$$\left(\frac{dR(t)}{dt}\right)^2 = -\left(1 - \frac{2M}{R(t)}\right)^3 \frac{1}{E^2} + \left(1 - \frac{2M}{R(t)}\right)^2\,, \tag{14}$$

where $R(t)$ is the location of the particle and the energy E is written by

$$E = \left(1 - \frac{2M}{R(t)}\right) \frac{dT(\tau)}{d\tau} \,. \tag{15}$$

We will also use

$$\frac{d^2R(t)}{dt^2} = -\frac{3}{E^2}\left(1 - \frac{2M}{R(t)}\right)^2 \frac{M}{R(t)^2} + 2\left(1 - \frac{2M}{R(t)}\right)\frac{M}{R(t)^2} \,, \tag{16}$$

to simplify equations. The tensor harmonics coefficients of the first order stress-energy tensor which are given in table A1,

$$\begin{aligned}
\mathcal{A}^{(1)}_{\ell m}(t,r) &= \mu \frac{E\,R(t)}{R(t)-2M}\left(\frac{dR(t)}{dt}\right)^2 \frac{1}{(r-2M)^2}\delta(r-R(t))Y^*_{\ell m}(0,0) \,, \\
\mathcal{A}^{(1)}_{0\,\ell m}(t,r) &= \mu \frac{E\,R(t)}{R(t)-2M}\frac{(r-2M)^2}{r^4}\delta(r-R(t))Y^*_{\ell m}(0,0) \,, \\
\mathcal{A}^{(1)}_{1\,\ell m}(t,r) &= \sqrt{2}i\mu \frac{E\,R(t)}{R(t)-2M}\frac{dR(t)}{dt}\frac{1}{r^2}\delta(r-R(t))Y^*_{\ell m}(0,0) \,.
\end{aligned} \tag{17}$$

The remaining coefficients are zero. Because of the symmetry of the problem, we have only to consider the even parity modes.

3.2. First order metric perturbations ($\ell \geq 2$)

We introduce the following wave-function for the even parity ($\ell \geq 2$) modes,

$$\begin{aligned}
\psi^{\rm even}_{\ell m}(t,r) = \frac{2\,r}{\ell(\ell+1)}\Bigg[K^{(1)RW}_{\ell m}(t,r) \\
+2\,\frac{(r-2\,M)}{(r\ell^2+r\ell-2\,r+6\,M)}\left(H^{(1)RW}_{2\,\ell m}(t,r) - r\frac{\partial}{\partial r}K^{(1)RW}_{\ell m}(t,r)\right)\Bigg] \,.
\end{aligned} \tag{18}$$

This function $\psi^{\rm even}_{\ell m}$ obeys the Zerilli equation,

$$\left(-\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial r^{*2}} - V^{\rm even}_\ell(r)\right)\psi^{\rm even}_{\ell m}(t,r) = S^{\rm even}_{\ell m}(t,r) \,, \tag{19}$$

where $r^* = r + 2M\log(r/2M - 1)$, the potential $V^{\rm even}_\ell$ is given in (A.9). The source $S^{\rm even}_{\ell m}$ is calculated as in (A.10). For the $\ell = 0$ and 1 modes, we will need a different treatment as described in section 3.4. (See also [48, 49].)

The reconstruction of the first order metric perturbations from this wave-function in the RW gauge have been given in (A.11). In the head on collision case, the metric perturbations in the RW gauge are C^0 (continuous across the particle). One can see this as follows. First, using the following linearized Hilbert-Einstein equations for each harmonics mode,

$$\frac{H^{(1)}_{0\,\ell m} - H^{(1)}_{2\,\ell m}}{2} = \frac{8\pi r^2 \mathcal{F}^{(1)}_{\ell m}}{\sqrt{\ell(\ell+1)(\ell-1)(\ell+2)/2}} \,, \tag{20}$$

and $\mathcal{F}^{(1)}_{\ell m} = 0$, we obtain $H^{(1)RW}_{2\,\ell m} = H^{(1)RW}_{0\,\ell m}$. Then, removing $H^{(1)RW}_{1\,\ell m}$ from two linearized Hilbert-Einstein equations ,

$$\frac{\partial}{\partial r}\left[\left(1-\frac{2M}{r}\right)H^{(1)RW}_{1\,\ell m}\right] - \frac{\partial}{\partial t}(H^{(1)RW}_{2\,\ell m} + K^{(1)RW}_{\ell m}) = \frac{8\pi i r}{\sqrt{\ell(\ell+1)/2}}\mathcal{B}^{(1)}_{0\,\ell m} \,, \tag{21}$$

$$-\frac{\partial H_{1\,\ell m}^{(1)RW}}{\partial t}+\left(1-\frac{2M}{r}\right)\frac{\partial}{\partial r}(H_{0\,\ell m}^{(1)RW}-K_{\ell m}^{(1)RW})+\frac{2M}{r^2}H_{0\,\ell m}^{(1)RW}$$
$$+\frac{1}{r}\left(1-\frac{M}{r}\right)(H_{2\,\ell m}^{(1)RW}-H_{0\,\ell m}^{(1)RW})=\frac{8\pi(r-2M)}{\sqrt{\ell(\ell+1)/2}}\mathcal{B}_{\ell m}^{(1)}\,, \tag{22}$$

we obtain the relation

$$\left[-\frac{\partial^2}{\partial t^2}+\left(1-\frac{2M}{r}\right)^2\frac{\partial^2}{\partial r^2}\right]H_{2\,\ell m}^{(1)RW}(t,r)=\left[\frac{\partial^2}{\partial t^2}+\left(1-\frac{2M}{r}\right)^2\frac{\partial^2}{\partial r^2}\right]K_{\ell m}^{(1)RW}(t,r)$$
$$+(\text{1st differential terms of } H_{2\,\ell m}^{(1)RW} \text{ and } K_{\ell m}^{(1)RW})\,, \tag{23}$$

where we have used $\mathcal{B}_{0\,\ell m}^{(1)}=\mathcal{B}_{\ell m}^{(1)}=0$. Therefore, we find that $H_{2\,\ell m}^{(1)RW}$ and $K_{\ell m}^{(1)RW}$ have the same differential behavior. Here, we note that the wave-function $\psi_{\ell m}^{\rm even}$ behaves as a step function around the particle location because of (19) (and (24)). Thus, it is found that $\partial_r K_{\ell m}^{(1)RW}\sim\theta(r-R(t))$ with use of (18). This means that $K_{\ell m}^{(1)RW}$ is C^0. From (22) with the above fact, $\partial_r H_{1\,\ell m}^{(1)RW}\sim\theta(r-R(t))$ is derived , i.e. $H_{1\,\ell m}^{(1)RW}$ is also C^0. (See [25].) We note that we can take up to second derivatives of the function $\psi_{\ell m}^{\rm even}$ with respect to t and r around the particle location as in (27).

In the next subsection, we treat only the $\ell=2$ mode which is the leading contribution in the first order perturbations. In this $\ell=2$ mode, we may consider only the $m=0$ mode because of $Y_{\ell m}(0,0)=0$ for $m\neq 0$ in (17).

3.3. $\ell=2$, $m=0$ mode

We focus here only on the $\ell=2$, $m=0$ mode. For this mode, the Zerilli equation in (19) becomes

$$\left[-\frac{\partial^2}{\partial t^2}+\frac{\partial^2}{\partial r^{*2}}-6\,\frac{(r-2\,M)(4\,r^3+4\,r^2M+6\,rM^2+3\,M^3)}{r^4(2\,r+3\,M)^2}\right]\psi_{20}^{\rm even}(t,r)$$
$$=-8\,\pi\,\frac{\mu\,\left(2\,R(t)^2-2\,R(t)E^2M+6\,R(t)M+M^2\right)\left(R(t)-2\,M\right)^2}{Er^3\left(2\,r+3\,M\right)^2}$$
$$\times Y_{20}^*(0,0)\,\delta\left(r-R(t)\right)+\frac{8\,\pi}{3}\,\frac{\mu\left(R(t)-2\,M\right)^3}{ER(t)^2\left(2\,R(t)+3\,M\right)}Y_{20}^*(0,0)\,\frac{d}{dr}\delta\left(r-R(t)\right)\,, \tag{24}$$

where we have used the formula

$$F(r)\frac{d}{dr}\delta'(r-R)=F(R)\frac{d}{dr}\delta(r-R)-\frac{d}{dr}F'(r)\Big|_{r=R}\delta(r-R)\,, \tag{25}$$

to simplify the source term.

Here, we decompose the wave-function in the following form,

$$\psi_{20}^{\rm even}(t,r)=\Psi_{20}^{out}(t,r)\theta(r-R(t))+\Psi_{20}^{in}(t,r)\theta(R(t)-r)$$
$$=\Psi_{20}^{\Theta}(t,r)\theta(r-R(t))+\Psi_{20}^{H}(t,r)\,;$$
$$\Psi_{20}^{\Theta}(t,r)=\Psi_{20}^{out}(t,r)-\Psi_{20}^{in}(t,r)\,,\quad \Psi_{20}^{H}(t,r)=\Psi_{20}^{in}(t,r)\,, \tag{26}$$

where Ψ_{20}^{out} and Ψ_{20}^{in}, i.e., also Ψ^{Θ} and Ψ^{H}, are homogeneous solution to the Zerilli equation. Using the fact that the first order metric perturbations are C^0, the following

six quantities can be derived

$$\Psi_{20}^{\Theta}(t,r)|_{r=R(t)} = \frac{8\pi}{3}\,\frac{\mu\,E\,R(t)}{2\,R(t)+3\,M}Y_{20}^{*}\,(0,0)\ ,$$
$$\frac{\partial}{\partial r}\Psi_{20}^{\Theta}(t,r)|_{r=R(t)} = 16\pi\,\frac{\mu\,E\ (R(t)^2+R(t)M+M^2)}{(2\,R(t)+3\,M)^2\,(2\,M-R(t))}Y_{20}^{*}\,(0,0)\ ,$$
$$\frac{\partial}{\partial t}\Psi_{20}^{\Theta}(t,r)|_{r=R(t)} = -8\pi\,\frac{\mu\,E\,R(t)}{(2\,R(t)+3\,M)\,(2\,M-R(t))}\,\frac{dR(t)}{dt}Y_{20}^{*}\,(0,0)\ ,$$
$$\frac{\partial^2}{\partial r^2}\Psi_{20}^{\Theta}(t,r)|_{r=R(t)} = -8\pi\,\frac{\mu\,E\,(7\,M^3-4\,R(t)M^2+4\,R(t)^2M-8\,R(t)^3)}{(2\,R(t)+3\,M)^3\,(2\,M-R(t))^2}Y_{20}^{*}\,(0,0)\ ,$$
$$\frac{\partial^2}{\partial r\partial t}\Psi_{20}^{\Theta}(t,r)|_{r=R(t)} = 8\pi\,\frac{\mu\,E\,(-2\,R(t)^2+6\,R(t)M+3\,M^2)}{(2\,R(t)+3\,M)^2\,(2\,M-R(t))^2}\,\frac{dR(t)}{dt}Y_{20}^{*}\,(0,0)\ ,$$
$$\frac{\partial^2}{\partial t^2}\Psi_{20}^{\Theta}(t,r)|_{r=R(t)} = -8\pi\,\frac{\mu\,EM}{(2\,R(t)+3\,M)^3\,R(t)^2}Y_{20}^{*}\,(0,0)\ . \tag{27}$$

In the above equations, first we take the derivatives, and then set $r = R(t)$. These quantities allow us to calculate the coefficients of the δ terms in the second order source.

3.4. $\ell = 0$ mode (Monopole perturbation)

Next, we consider the $\ell = 0$ perturbation ($\ell = 1$ modes can be completely eliminated in the center of mass coordinate system) which is present only in even parity. The metric perturbations and the gauge transformation are given as

$$\begin{aligned}\boldsymbol{h}_{00}^{(1)} &= \left(1-\frac{2M}{r}\right)H_{0\,00}^{(1)}(t,r)\boldsymbol{a}_{0\,00} - i\sqrt{2}H_{1\,00}^{(1)}(t,r)\boldsymbol{a}_{1\,00} \\ &+ \left(1-\frac{2M}{r}\right)^{-1}H_{2\,00}^{(1)}(t,r)\boldsymbol{a}_{00} + \sqrt{2}K_{00}^{(1)}(t,r)\boldsymbol{g}_{00},\end{aligned} \tag{28}$$

$$\xi_{\ell=0}^{(1)\mu} = \{V_0^{(1)}(t,r)Y_{00}(\theta,\phi), V_1^{(1)}(t,r)Y_{00}(\theta,\phi), 0, 0\}\,, \tag{29}$$

respectively. The metric perturbations transforms under the above gauge transformation from the G gauge to the G' gauge as

$$H_{0\,00}^{(1)G'}(t,r) = H_{0\,00}^{(1)G}(t,r) + 2\frac{\partial}{\partial t}V_0^{(1)G\to G'}(t,r) + \frac{2\,M}{r(r-2\,M)}V_1^{(1)G\to G'}(t,r)\,, \tag{30}$$

$$H_{1\,00}^{(1)G'}(t,r) = H_{1\,00}^{(1)G}(t,r) - \frac{r}{r-2\,M}\frac{\partial}{\partial t}V_1^{(1)G\to G'}(t,r) + \frac{r-2\,M}{r}\frac{\partial}{\partial r}V_0^{(1)G\to G'}(t,r)\,, \tag{31}$$

$$H_{2\,00}^{(1)G'}(t,r) = H_{2\,00}^{(1)G}(t,r) - 2\frac{\partial}{\partial r}V_1^{(1)G\to G'}(t,r) + \frac{2\,M}{r(r-2\,M)}V_1^{(1)G\to G'}(r)\,, \tag{32}$$

$$K_{00}^{(1)G'}(t,r) = K_{00}^{(1)G}(t,r) - \frac{2}{r}V_1^{(1)G\to G'}(t,r)\,. \tag{33}$$

Here, we can choose $V_0^{(1)G\to G'}$ and $V_1^{(1)G\to G'}$ so that $H_{1\,00}^{(1)Z} = K_{00}^{(1)Z} = 0$ where the suffix Z stands for the Zerilli gauge [41]. In this gauge, the two independent field equations are given by

$$\frac{\partial H_{2\,00}^{(1)Z}(t,r)}{\partial r} + \frac{1}{r-2M}H_{2\,00}^{(1)Z}(t,r) = \frac{8\pi r^3}{(r-2M)^2}\mathcal{A}_{0\,00}^{(1)}(t,r)\,, \tag{34}$$

$$\frac{\partial H_{0\,00}^{(1)Z}(t,r)}{\partial r} + \frac{1}{r-2M} H_{2\,00}^{(1)Z}(t,r) = -8\pi r \mathcal{A}_{00}^{(1)}(t,r)\,, \tag{35}$$

where $\mathcal{A}_{0\,00}^{(1)}$ and $\mathcal{A}_{00}^{(1)}$ are given in (17). We solve the first equation, and then obtain

$$H_{2\,00}^{(1)Z}(t,r) = 8\pi\mu E \frac{1}{r-2M} Y_{00}^{*}(0,0)\,\theta(r-R(t))\,. \tag{36}$$

Next, substituting the above quantity into the second equation, we obtain

$$H_{0\,00}^{(1)Z}(t,r) = 8\pi\mu E \left(\frac{1}{r-2M} - \frac{1}{R(t)-2M} - \frac{R(t)^2}{(R(t)-2M)^3}\left(\frac{dR(t)}{dt}\right)^2\right) \times Y_{00}^{*}(0,0)\,\theta(r-R(t))\,. \tag{37}$$

It is difficult however to construct the second order source from the above metric perturbations, since these are not C^0. We instead consider a new (singular) gauge transformation, chosen to make the metric perturbations C^0.

We consider the following gauge transformation. We call this the C gauge, where the first order metric perturbations is C^0 at the particle location

$$\begin{aligned}
V_0^{(1)Z\to C}(t,r) &= \frac{2\,\pi\,\mu\,Y_{00}^{*}(0,\,0)}{3\,E}\,\frac{(r-2\,M)\,(r+2\,M)\,(r^2+4\,M^2)}{r^4}\mathrm{INT}(t) \\
&- \frac{2\,\pi\,\mu\,E\;Y_{00}^{*}(0,\,0)}{3}\frac{(r-R(t))\,R(t)^3}{(R(t)-2\,M)^4\,r^4(r-2\,M)}\frac{dR(t)}{dt}(-r^2R(t)^3-R(t)^2r^3-r^4R(t) \\
&+10\,r^2MR(t)^2+10\,MR(t)r^3+8\,Mr^4+10\,rMR(t)^3-16\,rM^2R(t)^2-16\,M^2R(t)r^2 \\
&-16\,M^2r^3-R(t)^4r+10\,R(t)^5-62\,R(t)^4M+72\,R(t)^3M^2\big)\,\theta(r-R(t))\,,
\end{aligned} \tag{38}$$

$$V_1^{(1)Z\to C}(t,r) = 4\,\pi\,\mu\,EY_{00}^{*}(0,\,0)\,\frac{(r-2\,M)\,\,(r-R(t))\,R(t)^6}{r^6\,(R(t)-2\,M)^2}\,\theta(r-R(t))\,; \tag{39}$$

$$\begin{aligned}
\mathrm{INT}(t) &= \int\Big[\big(-54\,M^2+39\,R(t)M-50\,ME^2R(t)+12\,E^2R(t)^2 \\
&\qquad -6\,R(t)^2+16\,M^2E^2\big)\,/\,(R(t)-2\,M)^3\Big]dt \\
&= 6\,\frac{(-1+2\,E^2)\,E}{\sqrt{1-E^2}}\,\arctan\left\{\left(\frac{M}{1-E^2}-R(t)\right)\left[R(t)\left(\frac{2\,M}{1-E^2}-R(t)\right)\right]^{-1/2}\right\} \\
&+12\,E^2\ln\Bigg\{\Bigg(\frac{4\,M^2}{1-E^2}+\frac{2\,R(t)M}{1-E^2}-4\,R(t)M \\
&\qquad +\frac{4\,M\,E}{\sqrt{1-E^2}}\left[R(t)\left(\frac{2\,M}{1-E^2}-R(t)\right)\right]^{1/2}\Bigg)\,(M(R(t)-2\,M))^{-1}\Bigg\} \\
&+\frac{(13\,R(t)^2+48\,M^2-56\,R(t)M)\,E[E^2R(t)-R(t)+2\,M]^{1/2}\sqrt{R(t)}}{(R(t)-2\,M)^3}\,.
\end{aligned} \tag{40}$$

We then obtain the $\ell=0$ mode of the metric perturbations in this C gauge by using (30), (31), (32) and (33),

$$\begin{aligned}
H_{0\,00}^{(1)C}(t,r) &= \frac{4\,\pi\,\mu\,Y_{00}^{*}(0,\,0)}{3}\,\frac{(r-2\,M)\,(r+2\,M)\,(r^2+4\,M^2)}{E\,(R(t)-2\,M)^3\,r^4}(-54\,M^2+39\,R(t)M \\
&-50\,ME^2R(t)+12\,E^2R(t)^2-6\,R(t)^2+16\,M^2E^2)\,\theta(R(t)-r)
\end{aligned}$$

$$
\begin{aligned}
&-\frac{4\,\pi\,\mu\,Y_{00}^*(0,\,0)}{3}\,\frac{1}{(R(t)-2\,M)^2\,r^7\,(r-2\,M)\,E}(-792\,R(t)^3r^3M^3+792\,R(t)^2M^3r^4\\
&\quad+192\,R(t)r^3M^5-864\,r^3M^6+6\,R(t)^7MrE^2-398\,R(t)^5r^3M+256\,r^3M^6E^2\\
&\quad-128\,r^4M^5E^2-6\,R(t)^6Mr^2E^2+308\,R(t)^5Mr^3E^2-468\,R(t)^4r^3M^2E^2\\
&\quad-72\,R(t)^3r^3M^3E^2-144\,R(t)^2r^3M^4E^2-672\,R(t)r^3M^5E^2+336\,R(t)r^4M^4E^2\\
&\quad+50\,R(t)^6r^3-44\,R(t)^5r^4-24\,r^7M^2E^2-50\,R(t)^6E^2r^3+72\,R(t)^2r^4M^3E^2\\
&\quad+24\,R(t)r^7ME^2+476\,R(t)^3r^4M^2E^2+12\,R(t)^6rM^2E^2-282\,R(t)^4r^4ME^2\\
&\quad-6\,R(t)^2r^7E^2-96\,R(t)M^4r^4+359\,R(t)^4Mr^4-968\,R(t)^3M^2r^4+432\,r^4M^5\\
&\quad+1022\,R(t)^4r^3M^2-12\,R(t)^7M^2E^2+44\,R(t)^5r^4E^2\big)\,\theta(r-R(t))\,,\\
&H_{1\,00}^{(1)C}(t,r)=\frac{128\,\pi\,\mu\,Y_{00}^*(0,\,0)}{3\,E}\,\frac{M^4\,(r-2\,M)}{r^6}\,\mathrm{INT}(t)\\
&\quad+\frac{4\,\pi\,\mu\,E\,Y_{00}^*(0,\,0)}{3}\frac{(r-R(t))\;R(t)^5}{r^6\,(r-2\,M)\,(R(t)-2\,M)^4}\,\frac{d}{dt}R(t)(-72\,M^2r^2-288\,R(t)M^3\\
&\quad+68\,rR(t)M^2+60\,MR(t)r^2-132\,MR(t)^2r+144\,rM^3-40\,R(t)^3M+25\,R(t)^3r\\
&\quad+248\,R(t)^2M^2-12\,R(t)^2r^2)\,\theta(r-R(t))\,,\\
&H_{2\,00}^{(1)C}(t,r)=8\,\pi\,\mu\,EY_{00}^*(0,\,0)\frac{r-R(t)}{(r-2\,M)\,r^7\,(R(t)-2\,M)^2}(-21\,R(t)^6rM+22\,R(t)^6M^2\\
&\quad+5\,R(t)^6r^2-4\,R(t)^5Mr^2+R(t)^5r^3+4\,R(t)^5rM^2+4\,R(t)^4M^2r^2+R(t)^4r^4\\
&\quad-4\,R(t)^4Mr^3-4\,R(t)^3r^4M+4\,R(t)^3r^3M^2+R(t)^3r^5-4\,R(t)^2Mr^5\\
&\quad+4\,R(t)^2r^4M^2+R(t)^2r^6+4\,R(t)M^2r^5-4\,R(t)Mr^6+4\,M^2r^6\big)\;\theta(r-R(t))\,,\\
&K_{00}^{(1)C}(t,r)=-8\,\pi\,\mu\,EY_{00}^*(0,\,0)\frac{(r-2\,M)\;R(t)^6\,(r-R(t))}{r^7\,(R(t)-2\,M)^2}\,\theta(r-R(t))\,.
\end{aligned}
\tag{41}
$$

Note that all of the above metric perturbations are C^0 at the particle location and go to zero at $r=\infty$ and the horizon. Using the metric perturbations in the C gauge, we have discussed the second order source in [50]. But, with this method, we can not obtain the second order gravitational wave at infinity because this is not an asymptotic flat gauge.

We therefore must consider another treatment. The metric perturbations in (41) have the following asymptotic behavior for large r

$$
\begin{aligned}
&H_{0\,00}^{(1)C}(t,r)=\left(8\,\frac{\mu\,\pi\,E}{r}+16\,\frac{\mu\,\pi\,EM}{r^2}\right)Y_{00}^*(0,0)+\mathrm{O}(r^{-3})\,,\ H_{1\,00}^{(1)C}(t,r)=\mathrm{O}(r^{-4})\,,\\
&H_{2\,00}^{(1)C}(t,r)=\left(8\,\frac{\mu\,\pi\,E}{r}+16\,\frac{\mu\,\pi\,EM}{r^2}\right)Y_{00}^*(0,0)+\mathrm{O}(r^{-3})\,,\ K_{00}^{(1)C}(t,r)=\mathrm{O}(r^{-5})\,.
\end{aligned}
\tag{42}
$$

Hence, the metric up to the first perturbative order in the system becomes

$$
\begin{aligned}
ds^2&=-\left(1-\frac{2M}{r}\right)(1-H_{0\,00}^{(1)C}(t,r)Y_{00}(\theta,\phi))dt^2+H_{100}^{(1)C}(t,r)Y_{00}(\theta,\phi)dtdr\\
&\quad+\left(1-\frac{2M}{r}\right)^{-1}(1+H_{2\,00}^{(1)C}(t,r)Y_{00}(\theta,\phi))dr^2+r^2(1+K_{00}^{(1)C}(t,r)Y_{00}(\theta,\phi))d\Omega^2\\
&\sim-\left(1-\frac{2M+2\mu E}{r}\right)dt^2+\left(1-\frac{2M+2\mu E}{r}\right)^{-1}dr^2+r^2d\Omega^2\,.
\end{aligned}
\tag{43}
$$

We thus find that this perturbation is related to the mass increase of the system.

From the above analysis, we define the total mass $M_{tot} = M + \mu\, E$ as that of the system. This means that the particle's mass is absorbed in the background Schwarzschild mass. Then, the first order $\ell = 0$ renormalized metric perturbations become

$$\begin{aligned} H_{0\,00}^{(1)N}(t,r) &= H_{0\,00}^{(1)C}(t,r) - H_{0\,00}^{(1)M}(t,r)\,, \\ H_{2\,00}^{(1)N}(t,r) &= H_{2\,00}^{(1)C}(t,r) - H_{2\,00}^{(1)M}(t,r)\,, \\ H_{1\,00}^{(1)N}(t,r) &= H_{1\,00}^{(1)C}(t,r)\,, \quad K_{00}^{(1)N}(t,r) = K_{00}^{(1)C}(t,r)\,, \end{aligned} \tag{44}$$

where we have labeled this renormalized metric perturbations by N, and

$$H_{0\,00}^{(1)M}(t,r) = H_{2\,00}^{(1)M}(t,r) = 8\,\frac{\mu\,\pi\,E}{r-2\,M} Y_{00}^{*}(0,0)\,, \tag{45}$$

Now for the asymptotic behavior for large r, we have

$$H_{0\,00}^{(1)N}(t,r) = \mathrm{O}(r^{-4})\,, \quad H_{2\,00}^{(1)N}(t,r) = \mathrm{O}(r^{-5})\,. \tag{46}$$

Next, we will use the coefficients of the first order metric perturbations labeled by N in (44) to derive the second order source.

4. Second order perturbations in the Regge-Wheeler gauge

4.1. Second order Zerilli equation

Since the first order metric perturbations contain only $m = 0$ even parity modes, we can discuss the second order metric perturbations via the Zerilli equation. And we will choose the RW gauge condition. Here, we use a wave-function for the second perturbative order,

$$\chi_{20}^{\mathrm{Z}}(t,r) = \frac{1}{2\,r+3\,M}\left(r^2 \frac{\partial}{\partial t} K_{20}^{(2)RW}(t,r) - (r-2\,M) H_{1\,20}^{(2)RW}(t,r)\right)\,. \tag{47}$$

This is the same definition as in (A.7) for the first perturbative order. Here, we have considered the contribution from the $\ell = 0$ and 2 modes of the first perturbative order to the $\ell = 2$ mode of the second perturbative order since this gives the leading contribution to gravitational radiation. This Zerilli function satisfies the equation,

$$\begin{aligned} \hat{\mathcal{Z}}_2^{\mathrm{even}}\, \chi_{20}^{\mathrm{Z}}(t,r) &= \left[-\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial r^{*2}} - 6\,\frac{(r-2\,M)(4\,r^3+4\,r^2M+6\,rM^2+3\,M^3)}{r^4(2\,r+3\,M)^2}\right] \chi_{20}^{\mathrm{Z}}(t,r) \\ &= \mathcal{S}_{20}^{\mathrm{Z}}(t,r)\,; \end{aligned} \tag{48}$$

$$\begin{aligned} \mathcal{S}_{20}^{\mathrm{Z}}(t,r) &= \frac{8\,\pi\,\sqrt{3}\,(r-2\,M)^2}{3(2\,r+3\,M)} \frac{\partial}{\partial t}\mathcal{B}_{20}^{(2)}(t,r) + \frac{8\,\pi\,(r-2\,M)^2}{2\,r+3\,M} \frac{\partial}{\partial t}\mathcal{A}_{20}^{(2)}(t,r) \\ &\quad - \frac{8\,\sqrt{3}\,\pi\,(r-2\,M)}{3} \frac{\partial}{\partial t}\mathcal{F}_{20}^{(2)}(t,r) - \frac{4\,\sqrt{2}\,i\pi\,(r-2\,M)^2}{2\,r+3\,M} \frac{\partial}{\partial r}\mathcal{A}_{1\,20}^{(2)}(t,r) \\ &\quad - \frac{8\,\sqrt{2}\,i\pi\,(r-2\,M)\,(5\,r-3\,M)\,M}{r\,(2\,r+3\,M)^2}\mathcal{A}_{1\,20}^{(2)}(t,r) - \frac{8\,\sqrt{3}\,i\pi\,(r-2\,M)^2}{3(2\,r+3\,M)} \frac{\partial}{\partial r}\mathcal{B}_{0\,20}^{(2)}(t,r) \\ &\quad + \frac{32\,\sqrt{3}\,i\pi\,\,(3\,M^2+r^2)\,(r-2\,M)}{3\,r\,(2\,r+3\,M)^2}\mathcal{B}_{0\,20}^{(2)}(t,r)\,. \end{aligned} \tag{49}$$

The functions $\mathcal{B}_{20}^{(2)}$ etc. are derived from the effective stress-energy tensor on the left hand side of (11),

$$T_{\mu\nu}^{(2,eff)} = \left(T_{\mu\nu}^{(2,SF)} + T_{\mu\nu}^{(2,h)}\right) - \frac{1}{8\,\pi} G_{\mu\nu}^{(2)}[h^{(1)}, h^{(1)}]\,, \tag{50}$$

by the same tensor harmonics expansion as for the first perturbative order.

The second order metric perturbations from χ_{20}^{Z} in the RW gauge are given by

$$\frac{\partial}{\partial t} K_{20}^{(2)RW}\,(t,r) = 6\,\frac{(r^2 + rM + M^2)}{(2\,r + 3\,M)\,r^2} \chi_{20}^{\mathrm{Z}}\,(t,r) + \frac{(r - 2\,M)}{r} \frac{\partial}{\partial r} \chi_{20}^{\mathrm{Z}}\,(t,r)$$
$$+ \frac{4\,\sqrt{2}\,i\pi r\,(r - 2\,M)}{2\,r + 3\,M} \mathcal{A}_{1\,20}^{(2)}\,(t,r) + \frac{8\sqrt{3}\,i\pi r\,(r - 2\,M)}{3(2\,r + 3\,M)} \mathcal{B}_{0\,20}^{(2)}\,(t,r)\;, \tag{51}$$

$$\frac{\partial}{\partial t} H_{2\,20}^{(2)RW}\,(t,r) = r \frac{\partial^2}{\partial r \partial t} K_{20}^{(2)RW}\,(t,r) + 3\,\frac{M}{r^2} \chi_{20}^{\mathrm{Z}}\,(t,r) - \frac{(2\,r + 3\,M)}{r} \frac{\partial}{\partial r} \chi_{20}^{\mathrm{Z}}\,(t,r)$$
$$- \frac{8\sqrt{3}}{3}\,i\pi\,\mathcal{B}_{0\,20}^{(2)}\,(t,r)\,r\,, \tag{52}$$

$$H_{0\,20}^{(2)RW}\,(t,r) = H_{2\,20}^{(2)RW}\,(t,r) + \frac{8\sqrt{3}}{3}\,\pi\,\mathcal{F}_{20}^{(2)}\,(t,r)\,r^2\,. \tag{53}$$

Since the RW gauge is not asymptotically flat, we need to derive the second order metric perturbations in an asymptotic flat (AF) gauge to obtain the second order gravitational wave at spatial infinity. This is discussed in Appendix C.

Note that in (8). the delta function $\delta^{(4)}(x - z(\tau))$ in $T_{\mu\nu}^{(2,h)}$ includes an angular dependence, $\delta^{(2)}(\Omega - \Omega(\tau)) = \sum_{\ell m} Y_{\ell m}(\Omega) Y_{\ell m}^*(\Omega(\tau))$. We have considered only the contribution from the $\ell = 0$ and 2 modes of the first order perturbations. Consistently, we must use only the three components with the factors, $h_\alpha^{(1)\alpha}(\ell = 2) Y_{2m}(\Omega) Y_{2m}^*(\Omega(\tau))$, $h_\alpha^{(1)\alpha}(\ell = 2) Y_{0m}(\Omega) Y_{0m}^*(\Omega(\tau))$ and $h_\alpha^{(1)\alpha}(\ell = 0) Y_{2m}(\Omega) Y_{2m}^*(\Omega(\tau))$.

In the second order source, given in (49), we may wander if there is any δ^2 term which prevents us from making the second order calculation. The answer is "No". This is because, in the head-on collision case, the first order metric perturbations in the RW gauge are C^0: $G_{\mu\nu}^{(2)}[h^{(1)}, h^{(1)}]$ includes second derivatives and we need one more derivative to construct the second order source of (49). Here, $(h^{(1)})^2$ is $C^0 \times C^0$ and its third derivative yields $C^0 \times \delta'$ and $\theta \times \delta$ as the most singular terms. On the other hand, $T_{\mu\nu}^{(2,h)}$ includes only $C^0 \times \delta$ terms. Note also that there is no δ^2 terms coming from $T_{\mu\nu}^{(2,SF)}$, (which we ignored otherwise in this paper.) Using the result of [25, 26], we can include the contributions of $T_{\mu\nu}^{(2,SF)}$.

In the following subsection, we derive the second order source of the Zerilli equation in (49). The summary is given here. From (49), we obtain the second order source as

$$\mathcal{S}_{20}^{\mathrm{Z}}(t,r) = {}^{(2,2)}\mathcal{S}_{20}^{\mathrm{Z}}(t,r) + {}^{(0,2)}\mathcal{S}_{20}^{\mathrm{Z}}(t,r)\,, \tag{54}$$

where ${}^{(2,2)}\mathcal{S}_{20}^{\mathrm{Z}}$ and ${}^{(0,2)}\mathcal{S}_{20}^{\mathrm{Z}}$ are the contribution from $(\ell = 2)\cdot(\ell = 2)$ and $(\ell = 0)\cdot(\ell = 2)$, respectively. Note that while the above source term is locally integrable near the particle's location, some terms diverge as $r \to \infty$ or $2M$. This is not readily suitable for numerical calculations. We then consider some regularization for the asymptotic behavior. (See e.g. [31].) In order to obtain a second order source which behaves well

everywhere, we define a regularized Zerilli function by
$$\tilde{\chi}^{\rm Z}_{20}(t,r) = \chi^{\rm Z}_{20}(t,r) - \chi^{{\rm reg},(2,2)}_{20} - \chi^{{\rm reg},(0,2)}_{20} \,. \tag{55}$$
The best suited equation to solve the Zerilli equation numerically is then
$$\hat{\mathcal{Z}}^{\rm even}_2 \tilde{\chi}^{\rm Z}_{20}(t,r) = \mathcal{S}^{\rm Z,reg}_{20}(t,r)\,, \tag{56}$$
where the regular source $\mathcal{S}^{\rm Z,reg}_{20}$ is given by
$$\mathcal{S}^{\rm Z,reg}_{20}(t,r) = \left({}^{(2,2)}\mathcal{S}^{\rm Z}_{20}(t,r) - \hat{\mathcal{Z}}^{\rm even}_2 \chi^{{\rm reg},(2,2)}_{20}(t,r)\right) + \left({}^{(0,2)}\mathcal{S}^{\rm Z}_{20}(t,r) - \hat{\mathcal{Z}}^{\rm even}_2 \chi^{{\rm reg},(0,2)}_{20}(t,r)\right) . \tag{57}$$

4.2. Regularized second order source from $(\ell=2)\cdot(\ell=2)$

When we consider the asymptotic behavior of the second order source for large r, we use the retarded solution of the first order wave-function $\psi^{\rm even}_{\ell m}$ with the retarded time $t-r^*$, which we expand in inverse powers of r. The wave-function becomes
$$\psi^{\rm even}_{20}(t,r) = F_I'(t-r^*) + \frac{3}{r} F_I(t-r^*) + {\rm O}(r^{-2})\,, \tag{58}$$
where F_I is some function of $(t-r_*)$ and $F_I'(x) = dF(x)/dx$.

On the other hand, the wave-function is expanded near the horizon as,
$$\psi^{\rm even}_{\ell m}(t,r) = F_H'(t+r^*) + \frac{1}{4M} F_H(t+r^*) + \frac{27(r-2M)}{56M^2} F_H(t+r^*) + {\rm O}((r-2M)^2)\,. \tag{59}$$
Using these expansions , we derive a second order source which is regular everywhere.

In order to obtain a well behaved source for large values of r, we define a regularization function by
$$\chi^{{\rm reg},(2,2)}_{20}(t,r) = \frac{\sqrt{5}}{7\sqrt{\pi}} \frac{r^2}{2\,r+3\,M} \left(\frac{\partial}{\partial t} K^{(1)RW}_{20}(t,r)\right) K^{(1)RW}_{20}(t,r)\,. \tag{60}$$
Note that the regularization function is not unique. Therefore, this affects the formal expression of the second order gravitational wave as in (C.27). However, for any specific computation, there is no ambiguity in the physical final results.

Using the above regularization function, the regularized second order source ${}^{(2,2)}\mathcal{S}^{\rm Z,reg}_{20}$ from the $(\ell=2)\cdot(\ell=2)$ coupling is obtained
$$\begin{aligned} {}^{(2,2)}\mathcal{S}^{\rm Z,reg}_{20}(t,r) &= \left({}^{(2,2)}\mathcal{S}^{\rm Z}_{20}(t,r) - \hat{\mathcal{Z}}^{\rm even}_2 \chi^{{\rm reg},(2,2)}_{20}(t,r)\right) \\ &= {}^{G(2,2)}S^{out}_{20}(t,r)\theta(r-R(t)) + {}^{G(2,2)}S^{in}_{20}(t,r)\theta(R(t)-r) \\ &\quad + {}^{G(2,2)}S^{\delta}_{20}(t,r)\delta(r-R(t)) + {}^{G(2,2)}S^{\delta'}_{20}(t,r)\frac{d}{dr}\delta(r-R(t)) \\ &\quad + {}^{T(2,2)}S^{\delta}_{20}(t,r)\delta(r-R(t)) + {}^{T(2,2)}S^{\delta'}_{20}(t,r)\frac{d}{dr}\delta(r-R(t))\,, \end{aligned} \tag{61}$$
where the superscripts G and T denote the source terms which are derived from $G^{(2)}_{\mu\nu}[h^{(1)},h^{(1)}]$ and $T^{(2,h)}_{\mu\nu}$, respectively. The six source factors, ${}^{G(2,2)}S^{out}_{20}$, etc. are given in Appendix B. For the factors of the step functions in the above equation, ${}^{G(2,2)}S^{out}_{00}$ behaves as ${\rm O}(r^{-2})$ for large r, and ${}^{G(2,2)}S^{in}_{20}$ vanishes as ${\rm O}((r-2M)^1)$ at $r=2M$.

4.3. Regularized second order source from $(\ell = 0) \cdot (\ell = 2)$

The second order source derived from the $(\ell = 0) \cdot (\ell = 2)$ coupling needs regularization of its behavior near the horizon. Note that when we choose another gauge for the first order $\ell = 0$ mode, some regularization is also needed for the behavior for large r. To that end, we use the regularization function,

$$\begin{aligned}
\chi_{20}^{\mathrm{reg},(0,2)}(t,r) = & \frac{1}{5488\sqrt{\pi}} \frac{1061\,r - 2728\,M}{r} H_{2\,00}^{(1)N}(t,r) H_{2\,20}^{(1)RW}(t,r) \\
& - \frac{107}{2744\sqrt{\pi}} \frac{(r - 2\,M)\,M}{r} H_{2\,00}^{(1)N}(t,r) \frac{\partial}{\partial r} H_{2\,20}^{(1)RW}(t,r) \\
& + \frac{1}{2744\sqrt{\pi}} \frac{M\,(583\,r - 756\,M)}{r} H_{2\,00}^{(1)N}(t,r) \frac{\partial}{\partial t} H_{2\,20}^{(1)RW}(t,r) \,. \qquad (62)
\end{aligned}$$

Then, we obtain the second order regularized source from the $(\ell = 0) \cdot (\ell = 2)$ coupling

$$\begin{aligned}
{}^{(0,2)}\mathcal{S}_{20}^{\mathrm{Z}}(t,r) = & + \left({}^{(0,2)}\mathcal{S}_{20}^{\mathrm{Z}}(t,r) - \hat{\mathcal{Z}}_2^{\mathrm{even}} \chi_{20}^{\mathrm{reg},(0,2)}(t,r) \right) \\
= & {}^{G(0,2)}S_{00}^{out}(t,r)\theta(r - R(t)) + {}^{G(0,2)}S_{20}^{in}(t,r)\theta(R(t) - r) \\
& + {}^{G(0,2)}S_{20}^{\delta}(t,r)\delta(r - R(t)) + {}^{G(0,2)}S_{20}^{\delta'}(t,r)\frac{d}{dr}\delta(r - R(t)) \\
& + {}^{T(0,2)}S_{20}^{\delta}(t,r)\delta(r - R(t)) + {}^{T(0,2)}S_{20}^{\delta'}(t,r)\frac{d}{dr}\delta(r - R(t)) \\
& + {}^{T(2,0)}S_{20}^{\delta}(t,r)\delta(r - R(t)) + {}^{T(2,0)}S_{20}^{\delta'}(t,r)\frac{d}{dr}\delta(r - R(t)) \,. \qquad (63)
\end{aligned}$$

Here, we have written the contribution of $h_\alpha^{(1)\alpha}(\ell = 2) Y_{00}(\Omega) Y_{00}^*(\Omega(\tau))$ and $h_\alpha^{(1)\alpha}(\ell = 0) Y_{20}(\Omega) Y_{20}^*(\Omega(\tau))$ as ${}^{T(0,2)}S_{20}^{\delta}$ and ${}^{T(2,0)}S_{20}^{\delta}$, respectively. In the above equation, ${}^{G(0,2)}S_{00}^{out}$ behaves as $\mathrm{O}(r^{-2})$ for large r, and ${}^{G(0,2)}S_{20}^{in}$ vanishes as $\mathrm{O}((r - 2M)^1)$ at $r = 2M$.

5. Summary and Discussion

In this paper, we have completed, for the first time the program of self-consistent binary black hole second order perturbations in the small mass ratio limit. Our analysis applies to headon collision, but can be extended to arbitrary orbits if worked in the Lorenz gauge [51]. The first order perturbations of two black holes starting from rest at a finite distance have been solved in the frequency [52] and time [53] domains. Then the corrected trajectories (from the background geodesics) via the computation of the self-force have been computed in [25, 26]. Here, we obtained the regularized source for the second order Zerilli equation in the case of a particle falling radially into a Schwarzschild black hole. This is given by (57) with (61) and (63). Using this second order source, we are able to compute the second order contribution to gravitational radiation by numerical integration of the wave equation (56) and compare to full numerical simulations. The derivation of the second order gravitational wave from the second order Zerilli function is discussed in Appendix C.

A key point is to prove that there is no δ^2 term in the second order source. To show this, we have used the fact that the first order metric perturbations in the RW gauge are C^0. In general orbit case (including circular orbits), the first order metric perturbations are not C^0 in the RW gauge, but they are C^0 in the Lorenz gauge [51]. This later gauge choice favors the study of the second order perturbations for generic orbits.

To be fully consistent in the second perturbative order, we have to include the self-force contribution $T_{\mu\nu}^{(2,SF)}$ which is derived from the deviation from the background geodesic motion. The self-force for a head-on collision has been calculated in [25, 26], and in a circular orbit around a Schwarzschild black hole in [54, 55, 56], but have not been obtained in the general case yet.

Here, we have discussed only the $\ell = 2$ mode of the second perturbative order since this gives the leading contribution to gravitational radiation. There remain a question about the convergence of the second order metric perturbations, though. Based on the works by Rosenthal [57, 58, 59], we discuss this problem.

First, we consider a second order wave equation which is given by

$$\Box h^{(2)} = S_h^{(2)} \,, \tag{64}$$

where $\Box$ is the wave operator and the second order source $S_h^{(2)}$ is derived from the first order metric perturbations with a local behavior around the particle location as $h^{(1)} \sim \mathrm{O}(\epsilon^{-1})$, where the spatial separation $\epsilon = |\mathbf{x} - \mathbf{x_0}|$ with a particle location $\mathbf{x_0}$. In this discussion, we are not on the particle's world line but rather take a limit to the particle location. We need second derivatives to compute $S_h^{(2)}$, then the local behavior becomes $S_h^{(2)} \sim \mathrm{O}(\epsilon^{-4})$. For the above source, if we solve the wave equation by using a usual four dimensional Green's function method, this solution diverge everywhere [57]. Therefore, to obtain the finite solution, we need to remove the $\mathrm{O}(\epsilon^{-4})$ and $\mathrm{O}(\epsilon^{-3})$ terms from $S_h^{(2)}$. Here, we note that we can remove the $\mathrm{O}(\epsilon^{-3})$ terms by using a regular gauge transformation [58]. Since we consider the second order gravitational wave at infinity which is gauge invariant, the $\mathrm{O}(\epsilon^{-3})$ terms does not contribute. In practice, we use the gauge invariant wave-function in our calculation. Thus, we may discuss only the problematic $\mathrm{O}(\epsilon^{-4})$ terms.

Here, Rosenthal has already shown the most singular part of the second order metric perturbations as

$$h_s^{(2)} \sim \mathrm{O}(\epsilon^{-2}) \,. \tag{65}$$

This is a peculiar solution of $\Box h_s^{(2)} \sim \mathrm{O}(\epsilon^{-4})$. Using this solution, we rewrite the second order wave solution as

$$h_r^{(2)} = h^{(2)} - h_s^{(2)} \sim \Box^{-1}\left(S^{(2)} - \mathrm{O}(\epsilon^{-4})\right) \sim \Box^{-1}\left(\mathrm{O}(\epsilon^{-2})\right) \,, \tag{66}$$

The right hand side of the last line is finite after the integration by using the retarded Green's function as $\Box^{-1}$. The final result is,

$$h^{(2)} = h_s^{(2)} + h_r^{(2)} \,, \tag{67}$$

is the physical second-order gravitational perturbations [58]. Thus, if we construct a second order gauge invariant $\psi^{(2)}$ from the above metric perturbations, this is given by

$$\psi^{(2)} = \psi_s^{(2)} + \psi_r^{(2)} \,, \tag{68}$$

where $\psi_s^{(2)}$ and $\psi_r^{(2)}$ are derived from $h_s^{(2)}$ and $h_r^{(2)}$, respectively. As a result, the apparent divergence derived from the first consideration is only a gauge contribution.

Next, we consider the expansion in terms of tensor harmonics of the second order gauge invariant wave-function (which needs not to be same as the Regge-Wheeler or Zerilli function),

$$\Box_{\ell m} \psi_{\ell m}^{(2)} = S_{\ell m}^{(2)} \,. \tag{69}$$

In our situation, we solve the above equation by numerical integration. Formally, we can write the solution as $\psi_{\ell m}^{(2)} = \Box_{\ell m}^{-1} S_{\ell m}^{(2)}$. This $\Box_{\ell m}^{-1}$ means a numerical integration with an appropriate boundary condition. We may also use the retarded Green's function. Since the solution $\psi_{\ell m}^{(2)}$ is gauge invariant, this summation over the $(\ell,\, m)$ modes coincides with $\psi^{(2)}$ in (68). Hence, the summation of $\psi_{\ell m}^{(2)}$ over modes has a finite value, (except for the location of the particle.) In particular, the asymptotic behaviour for large r, where we need to compute gravitational radiation, is well defined.

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Appendix A. The Regge-Wheeler-Zerilli formalism in the time domain

In this paper, we have discussed the metric perturbations using the Regge-Wheeler-Zerilli formalism, with similar notation to that of Zerilli's paper [41]. There are some differences though that we summarize in this appendix.

For the first and second order metric perturbations, and stress-energy tensors, we expand $h_{\mu\nu}^{(i)}$ and $T_{\mu\nu}^{(i)}$ $(i = 1,\, 2)$ in tensor harmonics,

$$\begin{aligned}
\boldsymbol{h}^{(i)} = \sum_{\ell m} \Bigg[& \left(1 - \frac{2M}{r}\right) H_{0\,\ell m}^{(i)}(t,r) \boldsymbol{a}_{0\,\ell m} - i\sqrt{2} H_{1\,\ell m}^{(i)}(t,r) \boldsymbol{a}_{1\,\ell m} \\
& + \left(1 - \frac{2M}{r}\right)^{-1} H_{2\,\ell m}^{(i)}(t,r) \boldsymbol{a}_{\ell m} - \frac{i}{r}\sqrt{2\ell(\ell+1)} h_{0\,\ell m}^{(e)(i)}(t,r) \boldsymbol{b}_{0\,\ell m} \\
& + \frac{1}{r}\sqrt{2\ell(\ell+1)} h_{1\,\ell m}^{(e)(i)}(t,r) \boldsymbol{b}_{\ell m} + \left[\frac{1}{2}\ell(\ell+1)(\ell-1)(\ell+2)\right]^{1/2} G_{\ell m}^{(i)}(t,r) \boldsymbol{f}_{\ell m} \\
& + \left(\sqrt{2} K_{\ell m}^{(i)}(t,r) - \frac{\ell(\ell+1)}{\sqrt{2}} G_{\ell m}^{(i)}(t,r)\right) \boldsymbol{g}_{\ell m} - \frac{\sqrt{2\ell(\ell+1)}}{r} h_{0\,\ell m}^{(i)}(t,r) \boldsymbol{c}_{0\,\ell m}
\end{aligned}$$

$$+\frac{i\sqrt{2\ell(\ell+1)}}{r}h^{(i)}_{1\,\ell m}(t,r)\boldsymbol{c}_{\ell m}+\frac{[2\ell(\ell+1)(\ell-1)(\ell+2)]^{1/2}}{2r^2}h^{(i)}_{2\,\ell m}(t,r)\boldsymbol{d}_{\ell m}\Bigg]\,, \quad \text{(A.1)}$$

$$\boldsymbol{T}^{(i)}=\sum_{\ell m}\Big[\mathcal{A}^{(i)}_{0\,\ell m}\boldsymbol{a}_{0\,\ell m}+\mathcal{A}^{(i)}_{1\,\ell m}\boldsymbol{a}_{1\,\ell m}+\mathcal{A}^{(i)}_{\ell m}\boldsymbol{a}_{\ell m}+\mathcal{B}^{(i)}_{0\,\ell m}\boldsymbol{b}_{0\,\ell m}+\mathcal{B}^{(i)}_{\ell m}\boldsymbol{b}_{\ell m}$$
$$+\mathcal{Q}^{(i)}_{0\,\ell m}\boldsymbol{c}_{0\,\ell m}+\mathcal{Q}^{(i)}_{\ell m}\boldsymbol{c}_{\ell m}+\mathcal{D}^{(i)}_{\ell m}\boldsymbol{d}_{\ell m}+\mathcal{G}^{(i)}_{\ell m}\boldsymbol{g}_{\ell m}+\mathcal{F}^{(i)}_{\ell m}\boldsymbol{f}_{\ell m}\Big]\,, \quad \text{(A.2)}$$

where $\boldsymbol{a}_{0\,\ell m}$, $\boldsymbol{a}_{\ell m}\,,\cdots$ are tensor harmonics defined by (3.2-11) in [34].

The tensor harmonics can be classified into even and odd parities from the above expressions. Even parity modes are defined by the parity $(-1)^\ell$ under the transformation $(\theta,\phi)\rightarrow(\pi-\theta,\phi+\pi)$, while odd parity modes are by the parity $(-1)^{\ell+1}$. Using the orthogonality of the above tensor harmonics, we can derive the coefficient of the corresponding tensor harmonics expansion. For example,

$$\mathcal{A}^{(1)}_{0\,\ell m}(t,r)=\int\boldsymbol{T}^{(1)}\cdot\boldsymbol{a}^*_{0\,\ell m}\,d\Omega=\int\delta^{\mu\alpha}\delta^{\nu\beta}T^{(1)}_{\mu\nu}\,a^*_{0\,\ell m\,\alpha\beta}\,d\Omega\,, \quad \text{(A.3)}$$

where $*$ denotes the complex conjugate, $d\Omega=\sin\theta d\theta d\phi$ and $\delta^{\mu\alpha}$ has the component, $\delta^{\mu\alpha}=diag(1,\,1,\,1/r^2,\,1/(r^2\sin^2\theta))$.

We summarize the Regge-Wheeler-Zerilli formalism in the time domain. The linearized Hilbert-Einstein equation is given in (10). Next, we specify that the stress-energy tensor in the right hand side of the above equation be the one of a point particle moving along a geodesic;

$$\begin{aligned}T^{(1)\mu\nu}&=\mu\int_{-\infty}^{+\infty}\delta^{(4)}(x-z(\tau))\frac{dz^\mu}{d\tau}\frac{dz^\nu}{d\tau}d\tau\\&=\mu\,U^0\frac{dz^\mu}{dt}\frac{dz^\nu}{dt}\frac{\delta(r-R(t))}{r^2}\delta^{(2)}(\Omega-\Omega(t))\,,\end{aligned} \quad \text{(A.4)}$$

where we have used the following notations,

$$z^\mu=z^\mu(\tau)=\{T(\tau),R(\tau),\Theta(\tau),\Phi(\tau)\}\,,\quad U^0=\frac{dT(\tau)}{d\tau}\,, \quad \text{(A.5)}$$

for the particle's orbit. This stress-energy tensor is expressed in terms of the tensor harmonics as given in table A1 for the even parity modes.

Substituting (A.1) and (A.2) into (10), we obtain the linearized Hilbert-Einstein equation for each harmonic mode. Here, we use the RW gauge condition, $h^{(1)}_{2\,\ell m}=0$ for the odd part and $h^{(e)(1)}_{0\,\ell m}=h^{(e)(1)}_{1\,\ell m}=G^{(1)}_{\ell m}=0$ for the even part. From these ten linearized equations, we derive the Regge-Wheeler-Zerilli equations and construct the metric perturbations from the Regge-Wheeler-Zerilli functions in the RW gauge. In the following, we focus on the even parity modes.

We introduce the following function,

$$\psi^{\rm even}_{\ell m}(t,r)=\frac{2\,r}{\ell(\ell+1)}\Big[K^{(1)}_{\ell m}(t,r)$$
$$+2\,\frac{(r-2\,M)}{(r\ell^2+r\ell-2\,r+6\,M)}\left(H^{(1)}_{2\,\ell m}(t,r)-r\,\frac{\partial}{\partial r}K^{(1)}_{\ell m}(t,r)\right)\Big]\,. \quad \text{(A.6)}$$

Table A1. The stress-energy tensor for the even parity modes.

$$\mathcal{A}^{(1)}_{\ell m}(t,r) = \mu U^0 \left(\frac{dR}{dt}\right)^2 (r-2M)^{-2}\delta(r-R(t))Y^*_{\ell m}(\Omega(t))$$
$$\mathcal{A}^{(1)}_{0\,\ell m}(t,r) = \mu U^0 \left(1-\frac{2M}{r}\right)^2 r^{-2}\delta(r-R(t))Y^*_{\ell m}(\Omega(t))$$
$$\mathcal{A}^{(1)}_{1\,\ell m}(t,r) = \sqrt{2}i\mu U^0 \frac{dR}{dt} r^{-2}\delta(r-R(t))Y^*_{\ell m}(\Omega(t))$$
$$\mathcal{B}^{(1)}_{0\,\ell m}(t,r) = [\ell(\ell+1)/2]^{-1/2} i\mu U^0 \left(1-\frac{2M}{r}\right) r^{-1}\delta(r-R(t))dY^*_{\ell m}(\Omega(t))/dt$$
$$\mathcal{B}^{(1)}_{\ell m}(t,r) = [\ell(\ell+1)/2]^{-1/2} \mu U^0 (r-2M)^{-1}\frac{dR}{dt}\delta(r-R(t))dY^*_{\ell m}(\Omega(t))/dt$$
$$\mathcal{F}^{(1)}_{\ell m}(t,r) = [\ell(\ell+1)(\ell-1)(\ell+2)/2]^{-1/2}\mu U^0\delta(r-R(t)) \times\left(\frac{d\Phi}{dt}\frac{d\Theta}{dt}X^*_{\ell m}[\Omega(t)] + \frac{1}{2}\left[(\frac{d\Theta}{dt})^2 - \sin^2\Theta(\frac{d\Phi}{dt})^2\right]W^*_{\ell m}[\Omega(t)]\right)$$
$$\mathcal{G}^{(1)}_{\ell m}(t,r) = \frac{\mu U^0}{\sqrt{2}}\delta(r-R(t))\left[(\frac{d\Theta}{dt})^2 + \sin^2\Theta(\frac{d\Phi}{dt})^2\right]Y^*_{\ell m}(\Omega(t))$$

This function is related to Zerilli's even parity function $\psi^{\rm Z,even}_{\ell m}$ as

$$\psi^{\rm Z,even}_{\ell m}(t,r) = \frac{2}{r\ell^2 + r\ell - 2\,r + 6\,M}\left(r^2\frac{\partial}{\partial t}K^{(1)}_{\ell m}(t,r) - (r-2\,M)H^{(1)}_{1\,\ell m}(t,r)\right)$$
$$= \frac{\partial}{\partial t}\,\psi^{\rm even}_{\ell m}(t,r) - \frac{16\sqrt{2}\,\pi i\, r^2(r-2\,M)}{\ell(\ell+1)(r\ell^2+r\ell-2\,r+6\,M)}\mathcal{A}^{(1)}_{1\,\ell m}(t,r)\,. \qquad (A.7)$$

The function $\psi^{\rm even}_{\ell m}$ obeys the Zerilli equation,

$$\left[-\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial r^{*2}} - V^{\rm even}_\ell(r)\right]\psi^{\rm even}_{\ell m}(t,r) = S^{\rm even}_{\ell m}(t,r)\,, \qquad (A.8)$$

where $r^* = r + 2M\ln(r/2M-1)$ and

$$V^{\rm even}_\ell(r) = \frac{r-2M}{r^4(r\ell^2+r\ell-2\,r+6\,M)^2}\big(\ell(\ell+1)(\ell+2)^2(\ell-1)^2r^3 +6\,M(\ell+2)^2(\ell-1)^2r^2 + 36\,M^2(\ell+2)(\ell-1)r + 72\,M^3\big)\,, \qquad (A.9)$$

and the source term is given by

$$S^{\rm even}_{\ell m}(t,r) = \frac{16\,\pi\,(r-2\,M)^2(r\ell^2+r\ell-4\,r+2\,M)}{\ell(\ell+1)(r\ell^2+r\ell-2\,r+6\,M)r}\mathcal{A}^{(1)}_{\ell m}(t,r)$$
$$-\frac{16\sqrt{2}\,\pi(r-2\,M)}{\sqrt{\ell(\ell+1)(\ell-1)(\ell+2)}}\mathcal{F}^{(1)}_{\ell m}(t,r) + \frac{32\,\pi\,(r-2\,M)^2\sqrt{2}}{(r\ell^2+r\ell-2\,r+6\,M)\sqrt{\ell(\ell+1)}}\mathcal{B}^{(1)}_{\ell m}(t,r)$$
$$-\frac{32\,\pi(r-2\,M)^3}{(r\ell^2+r\ell-2\,r+6\,M)\ell(\ell+1)}\frac{\partial}{\partial r}\mathcal{A}^{(1)}_{\ell m}(t,r) - \Big\{16\,\pi\,r(\ell^4r^2 + 2\,r^2\ell^3 - 5\,r^2\ell^2 +16\,r\ell^2M - 6\,r^2\ell + 16\,r\ell M + 8\,r^2 - 68\,rM + 108\,M^2)/[(\ell+1)\ell(r\ell^2+r\ell -2\,r+6\,M)^2]\Big\}\mathcal{A}^{(1)}_{0\,\ell m}(t,r) + \frac{32\,\pi(r-2\,M)r^2}{(r\ell^2+r\ell-2\,r+6\,M)\ell(\ell+1)}\frac{\partial}{\partial r}\mathcal{A}^{(1)}_{0\,\ell m}(t,r)$$
$$+\frac{32\sqrt{2}\,\pi(r-2\,M)^2}{(r\ell^2+r\ell-2\,r+6\,M)\ell(\ell+1)}\mathcal{G}^{(1)}_{\ell m}(t,r)\,. \qquad (A.10)$$

Here $T^{(1);\nu}_{\mu\nu} = 0$ have been used to simplify the source term. Using the function $\psi^{\rm even}_{\ell m}$, the four coefficients for the metric perturbations in the RW gauge are expressed as

$$K^{(1)RW}_{\ell m}(t,r) = \frac{1}{2}\Big\{(\ell^4 r^2 + 2\,r^2\ell^3 - r^2\ell^2 + 6\,r\ell^2 M - 2\,r^2\ell + 6\,r\ell M \\ -12\,rM + 24\,M^2)/\left[(r\ell^2 + r\ell - 2\,r + 6\,M)r^2\right]\Big\}\,\psi^{(\rm even)}_{\ell m}(t,r) \\ +\frac{(r-2\,M)}{r}\frac{\partial}{\partial r}\psi^{(\rm even)}_{\ell m}(t,r) - \frac{32\,\pi r^3}{\ell(\ell+1)(r\ell^2 + r\ell - 2\,r + 6\,M)}\,\mathcal{A}^{(1)}_{0\,\ell m}(t,r)\,,$$

$$H^{(1)RW}_{2\,\ell m}(t,r) = -\frac{1}{2}\,\frac{(r\ell^2 + r\ell - 2\,r + 6\,M)}{r-2\,M}\,K^{(1)RW}_{\ell m}(t,r) + r\frac{\partial}{\partial r}K^{(1)RW}_{\ell m}(t,r) \\ +\frac{1}{4}\,\frac{\ell(\ell+1)(r\ell^2 + r\ell - 2\,r + 6\,M)}{(r-2\,M)r}\,\psi^{(\rm even)}_{\ell m}(t,r)\,,$$

$$H^{(1)RW}_{0\,\ell m}(t,r) = H^{(1)RW}_{2\,\ell m}(t,r) + 16\,\frac{\pi r^2\sqrt{2}}{\sqrt{\ell(\ell+1)(\ell-1)(\ell+2)}}\,\mathcal{F}^{(1)}_{\ell m}(t,r)\,,$$

$$H^{(1)RW}_{1\,\ell m}(t,r) = 2\,\frac{r(r-3\,M)}{(r-2\,M)\ell(\ell+1)}\,\frac{\partial}{\partial t}K^{(1)RW}_{\ell m}(t,r) + 2\,\frac{r^2}{\ell(\ell+1)}\frac{\partial^2}{\partial t\partial r}K^{(1)RW}_{\ell m}(t,r) \\ -2\,\frac{r}{\ell(\ell+1)}\,\frac{\partial}{\partial t}H^{(1)RW}_{2\,\ell m}(t,r) + \frac{8\,i\pi\sqrt{2}r^2}{\ell(\ell+1)}\,\mathcal{A}^{(1)}_{1\,\ell m}(t,r)\,. \qquad \text{(A.11)}$$

Appendix B. Second order source terms

In this appendix, we show the regularized source for the second order Zerilli equation in (56) from the contribution of the $(\ell=2)\cdot(\ell=2)$ coupling explicitly. We can also derive the source from the $(\ell=0)\cdot(\ell=2)$ coupling in the same way, but the expression is so long that we do not have the space to show it here. The regularized second order source (61) from the $(\ell=2)\cdot(\ell=2)$ coupling has the following six factors

$$ {}^{G(2,2)}S^{out}_{20}(t,r) = \frac{2}{7}\sqrt{5}\,(r-2\,M)\left[-\frac{3\,(8\,r^2 + 12\,M\,r + 7\,M^2)\,(r-2\,M)}{r^2\,(2\,r+3\,M)}\,\Psi^{out}_{20}(t,r)\right. \\ \times\left(\frac{\partial^2}{\partial t\,\partial r}\,\Psi^{out}_{20}(t,r)\right) + \frac{3\,(2\,r^2 - M^2)\,(r-2\,M)}{r\,(2\,r+3\,M)}\,\left(\frac{\partial^2}{\partial r^2}\,\Psi^{out}_{20}(t,r)\right) \\ \times\left(\frac{\partial}{\partial t}\,\Psi^{out}_{20}(t,r)\right) - \frac{(18\,r^3 - 4\,r^2\,M - 33\,r\,M^2 - 48\,M^3)}{r^2\,(2\,r+3\,M)}\,\left(\frac{\partial}{\partial t}\,\Psi^{out}_{20}(t,r)\right) \\ \times\left(\frac{\partial}{\partial r}\,\Psi^{out}_{20}(t,r)\right) - (r-2\,M)^2\,\left(\frac{\partial^3}{\partial t\,\partial r^2}\,\Psi^{out}_{20}(t,r)\right)\,\left(\frac{\partial}{\partial r}\,\Psi^{out}_{20}(t,r)\right) - 3\,(112\,r^5 \\ +480\,r^4\,M + 692\,r^3\,M^2 + 762\,r^2\,M^3 + 441\,r\,M^4 + 144\,M^5)\,\left(\frac{\partial}{\partial t}\,\Psi^{out}_{20}(t,r)\right) \\ \times\Psi^{out}_{20}(t,r)\,/(r^3\,(2\,r+3\,M)^3) - \frac{(r-2\,M)\,(3\,r-7\,M)}{r}\,\left(\frac{\partial}{\partial r}\,\Psi^{out}_{20}(t,r)\right) \\ \times\left(\frac{\partial^2}{\partial t\,\partial r}\,\Psi^{out}_{20}(t,r)\right) + \frac{3\,(r-2\,M)^2\,M}{r\,(2\,r+3\,M)}\,\left(\frac{\partial^3}{\partial t\,\partial r^2}\,\Psi^{out}_{20}(t,r)\right)\,\Psi^{out}_{20}(t,r)$$

$$+(r-2\,M)^2\,\left(\frac{\partial^3}{\partial r^3}\,\Psi^{out}_{20}(t,r)\right)\,\left(\frac{\partial}{\partial t}\,\Psi^{out}_{20}(t,r)\right)\Bigg]\,/[\sqrt{\pi}\,(2\,r+3\,M)\,r^2]\,, \tag{B.1}$$

$$\begin{aligned}
{}^{G(2,2)}S^{in}_{20}(t,r) &= \frac{2}{7}\sqrt{5}\,(r-2\,M)\left[\frac{3\,(r-2\,M)^2\,M}{r\,(2\,r+3\,M)}\,\Psi^{in}_{20}(t,r)\,\left(\frac{\partial^3}{\partial t\,\partial r^2}\,\Psi^{in}_{20}(t,r)\right)\right.\\
&-(r-2\,M)^2\,\left(\frac{\partial}{\partial r}\,\Psi^{in}_{20}(t,r)\right)\,\left(\frac{\partial^3}{\partial t\,\partial r^2}\,\Psi^{in}_{20}(t,r)\right)-(18\,r^3-4\,r^2\,M-33\,r\,M^2\\
&\quad-48\,M^3)/(r^2\,(2\,r+3\,M))\,\left(\frac{\partial}{\partial t}\,\Psi^{in}_{20}(t,r)\right)\,\left(\frac{\partial}{\partial r}\,\Psi^{in}_{20}(t,r)\right)\\
&-\frac{(r-2\,M)\,(3\,r-7\,M)}{r}\,\left(\frac{\partial^2}{\partial t\,\partial r}\,\Psi^{in}_{20}(t,r)\right)\,\left(\frac{\partial}{\partial r}\,\Psi^{in}_{20}(t,r)\right)-3\,(112\,r^5+480\,r^4\,M\\
&\quad+692\,r^3\,M^2+762\,r^2\,M^3+441\,r\,M^4+144\,M^5)/(r^3\,(2\,r+3\,M)^3)\Psi^{in}_{20}(t,r)\\
&\quad\times\left(\frac{\partial}{\partial t}\,\Psi^{in}_{20}(t,r)\right)\,-\,\frac{3\,(8\,r^2+12\,M\,r+7\,M^2)\,(r-2\,M)}{r^2\,(2\,r+3\,M)}\,\left(\frac{\partial^2}{\partial t\,\partial r}\,\Psi^{in}_{20}(t,r)\right)\\
&\quad\times\Psi^{in}_{20}(t,r)+\frac{3\,(2\,r^2-M^2)\,(r-2\,M)}{r\,(2\,r+3\,M)}\,\left(\frac{\partial^2}{\partial r^2}\,\Psi^{in}_{20}(t,r)\right)\,\left(\frac{\partial}{\partial t}\,\Psi^{in}_{20}(t,r)\right)\\
&\left.+(r-2\,M)^2\,\left(\frac{\partial^3}{\partial r^3}\,\Psi^{in}_{20}(t,r)\right)\,\left(\frac{\partial}{\partial t}\,\Psi^{in}_{20}(t,r)\right)\right]\,/[\sqrt{\pi}(2\,r+3\,M)\,r^2]\,,
\end{aligned} \tag{B.2}$$

$$\begin{aligned}
{}^{G(2,2)}S^{\delta}_{20}(t,r) &= \frac{4}{7}\sqrt{5}\,\sqrt{\pi}\,\mu\,Y^*_{20}(0,0)\Bigg[-\frac{dR(t)}{dt}\,(-R(t)+2\,M)^2\,(2\,M+3\,E^2\,R(t)-R(t))\\
&\quad\times\left(\frac{\partial^3}{\partial r^3}\,\Psi^{in}_{20}(t,r)\right)\Bigg|_{r\,=\,R(t)}-2\frac{dR(t)}{dt}(-38\,R(t)\,M^3-48\,M^4-40\,R(t)^2\,E^2\,M^2\\
&\quad+282\,M^3\,R(t)\,E^2+288\,M^4\,E^2+171\,R(t)^2\,M^2+66\,M\,E^2\,R(t)^3+12\,R(t)^4\\
&\quad-94\,R(t)^3\,M-64\,R(t)^4\,E^2)\times\left(\frac{\partial}{\partial r}\,\Psi^{in}_{20}(t,r)\right)\Bigg|_{r\,=\,R(t)}\Big/(R(t)\,(2\,R(t)+3\,M)^2)\\
&+6\,\frac{dR(t)}{dt}(-48\,E^2\,R(t)^5+8\,R(t)^5+288\,M^5\,E^2+546\,R(t)\,M^4\,E^2+52\,R(t)^4\,M\,E^2\\
&\quad-60\,R(t)^4\,M-41\,R(t)^2\,M^3+288\,R(t)^2\,M^3\,E^2+104\,R(t)^3\,M^2+36\,M^5\\
&\quad-8\,R(t)^3\,M^2\,E^2)\Psi^{in}_{20}(t,\,R(t))\Big/(R(t)^2\,(2\,R(t)+3\,M)^3)-(-R(t)+2\,M)\\
&\quad\times(18\,M^3+5\,R(t)\,M^2+6\,M^2\,R(t)\,E^2+14\,M\,R(t)^2\,E^2-43\,R(t)^2\,M+18\,R(t)^3\\
&\quad-12\,R(t)^3\,E^2)\left(\frac{\partial^2}{\partial r\,\partial t}\,\Psi^{in}_{20}(t,r)\right)\Bigg|_{r\,=\,R(t)}\Big/(R(t)\,(2\,R(t)+3\,M))+(216\,M^5\\
&\quad+162\,R(t)\,M^4-72\,R(t)\,M^4\,E^2-150\,R(t)^2\,M^3\,E^2+165\,R(t)^2\,M^3-4\,R(t)^5\\
&\quad-386\,R(t)^3\,M^2+36\,R(t)^3\,M^2\,E^2-108\,R(t)^4\,M\,E^2+126\,R(t)^4\,M\\
&\quad+56\,E^2\,R(t)^5)\left(\frac{\partial}{\partial t}\,\Psi^{in}_{20}(t,r)\right)\Bigg|_{r\,=\,R(t)}\Big/(R(t)^2\,(2\,R(t)+3\,M)^2)\\
&-\frac{1}{2}\,(-R(t)+2\,M)^2\,(10\,M+6\,E^2\,R(t)-5\,R(t))\left(\frac{\partial^3}{\partial r^2\,\partial t}\,\Psi^{in}_{20}(t,r)\right)\Bigg|_{r\,=\,R(t)}\\
&+(-R(t)+2\,M)\,\frac{dR(t)}{dt}(48\,M^3+26\,R(t)\,M^2+66\,M^2\,R(t)\,E^2+34\,M\,R(t)^2\,E^2
\end{aligned}$$

$$-13\,R(t)^2\,M+12\,R(t)^3\,E^2-6\,R(t)^3)\left(\frac{\partial^2}{\partial r^2}\,\Psi^{in}_{20}(t,r)\right)\Bigg|_{r\,=\,R(t)}$$
$$\Big/(R(t)\,(2\,R(t)+3\,M))-\frac{dR(t)}{dt}\,(-R(t)+2\,M)^2\,(2\,M+3\,E^2\,R(t)-R(t))$$
$$\times\left(\frac{\partial^3}{\partial r^3}\,\Psi^{out}_{20}(t,r)\right)\Bigg|_{r\,=\,R(t)}-\frac{1}{2}\,(-R(t)+2\,M)^2\,(10\,M+6\,E^2\,R(t)-5\,R(t))$$
$$\times\left(\frac{\partial^3}{\partial r^2\,\partial t}\,\Psi^{out}_{20}(t,r)\right)\Bigg|_{r\,=\,R(t)}-4\,\pi\frac{dR(t)}{dt}\,\mu\,Y^*_{20}(0,0)(810\,M^6+1113\,M^5\,R(t)$$
$$+972\,M^5\,R(t)\,E^2+1710\,M^4\,R(t)^2\,E^2+1161\,M^4\,R(t)^2+2124\,M^3\,R(t)^3\,E^2$$
$$-2352\,M^3\,R(t)^3+328\,M^2\,R(t)^4-1296\,M^2\,R(t)^4\,E^2-96\,M\,R(t)^5\,E^2$$
$$+120\,M\,R(t)^5+32\,R(t)^6-288\,E^2\,R(t)^6)E$$
$$\Big/(R(t)\,(2\,R(t)+3\,M)^4\,(-R(t)+2\,M))\Big]\Big/[E R(t)^2\,(2\,R(t)+3\,M)]\,, \tag{B.3}$$

$$^{G(2,2)}S^{\delta'}_{20}(t,r)=\frac{4}{7}\mu\,\sqrt{5}\,\sqrt{\pi}\,Y^*_{20}(0,0)\Big[-(-R(t)+2\,M)(-10\,M^2+6\,E^2\,M^2+17\,R(t)\,M$$
$$-4\,M\,E^2\,R(t)-6\,R(t)^2+4\,E^2\,R(t)^2)\,\frac{dR(t)}{dt}\left(\frac{\partial}{\partial r}\,\Psi^{in}_{20}(t,r)\right)\Bigg|_{r\,=\,R(t)}$$
$$\Big/(2\,R(t)+3\,M)+6\,(-R(t)+2\,M)(3\,E^2\,M^3+2\,M^3+6\,M^2\,R(t)\,E^2-3\,R(t)\,M^2$$
$$+4\,M\,R(t)^2\,E^2+5\,R(t)^2\,M+4\,R(t)^3\,E^2-2\,R(t)^3)\frac{dR(t)}{dt}\Psi^{in}_{20}(t,\,R(t))$$
$$\Big/(R(t)\,(2\,R(t)+3\,M)^2)+(-R(t)+2\,M)^2\,(2\,E^2\,R(t)-3\,R(t)+6\,M)$$
$$\times\left(\frac{\partial^2}{\partial r\,\partial t}\,\Psi^{in}_{20}(t,r)\right)\Bigg|_{r\,=\,R(t)}+2\,\frac{dR(t)}{dt}\,(-R(t)+2\,M)^2\,(E^2\,R(t)-R(t)+2\,M)$$
$$\times\left(\frac{\partial^2}{\partial r^2}\,\Psi^{in}_{20}(t,r)\right)\Bigg|_{r\,=\,R(t)}+(-R(t)+2\,M)\,(3\,M^2+6\,R(t)\,M-2\,R(t)^2)$$
$$\times\,(2\,E^2\,R(t)-3\,R(t)+6\,M)\left(\frac{\partial}{\partial t}\,\Psi^{in}_{20}(t,r)\right)\Bigg|_{r\,=\,R(t)}\Big/(R(t)\,(2\,R(t)+3\,M))\Big]$$
$$\Big/[E\,R(t)^2\,(2\,R(t)+3\,M)]\,, \tag{B.4}$$

$$^{T(2,2)}S^{\delta}_{20}(t,r)=\frac{8}{7}\sqrt{5}\,\sqrt{\pi}\,\mu\,(R(t)-2\,M)\,Y^*_{20}(0,0)\,\Big[-2(24\,M^3+10\,R(t)\,M^2$$
$$+6\,M^2\,R(t)\,E^2+9\,M\,R(t)^2\,E^2-7\,R(t)^2\,M+R(t)^3\,E^2-2\,R(t)^3)\frac{dR(t)}{dt}$$
$$\times\left(\frac{\partial}{\partial r}\,\Psi^{in}_{20}(t,r)\right)\Bigg|_{r\,=\,R(t)}\Big/(R(t)\,(2\,R(t)+3\,M))+6\,\frac{dR(t)}{dt}(24\,M^4$$
$$+6\,M^3\,R(t)\,E^2+36\,R(t)\,M^3+21\,R(t)^2\,M^2+13\,R(t)^2\,E^2\,M^2+10\,R(t)^3\,M$$
$$+8\,M\,E^2\,R(t)^3+6\,R(t)^4\,E^2-4\,R(t)^4)\Psi^{in}_{20}(t,\,R(t))\Big/(R(t)^2\,(2\,R(t)+3\,M)^2)$$
$$+(E^2\,R(t)-R(t)+2\,M)\,(-R(t)+2\,M)\,\left(\frac{\partial^2}{\partial r\,\partial t}\,\Psi^{in}_{20}(t,r)\right)\Bigg|_{r\,=\,R(t)}$$

$$+(E^2\,R(t)-R(t)+2\,M)\,\frac{dR(t)}{dt}\,(-R(t)+2\,M)\left(\frac{\partial^2}{\partial r^2}\,\Psi_{20}^{in}(t,r)\right)\Bigg|\,r=R(t)$$
$$-\frac{6\,(E^2\,R(t)-R(t)+2\,M)\,(R(t)^2+R(t)\,M+M^2)}{R(t)\,(2\,R(t)+3\,M)}\left(\frac{\partial}{\partial t}\,\Psi_{20}^{in}(t,r)\right)\Bigg|\,r=R(t)\Bigg]$$
$$\Big/[ER(t)^3\,(2\,R(t)+3\,M)]\,, \tag{B.5}$$

$${}^{T(2,2)}S_{20}^{\delta'}(t,r)=\frac{8}{7}\sqrt{5}\,\sqrt{\pi}\,\mu\,(R(t)-2\,M)^2\,Y_{20}^*(0,0)\left[\frac{dR(t)}{dt}\,(-R(t)+2\,M)\right.$$
$$\times\left(\frac{\partial}{\partial r}\,\Psi_{20}^{in}(t,r)\right)\Bigg|\,r=R(t)-\frac{dR(t)}{dt}\frac{6\;(R(t)^2+R(t)\,M+M^2)}{R(t)\,(2\,R(t)+3\,M)}\,\Psi_{20}^{in}(t,\,R(t))\Bigg]$$
$$\Big/(E\,R(t)^3\,(2\,R(t)+3\,M))\,. \tag{B.6}$$

Here, we have used (27) for Ψ_{20}^{Θ} and its derivatives at the particle trajectory. And for the homogeneous solutions, Ψ_{20}^{out} and Ψ_{20}^{in} have been used to write the above source terms.

Appendix C. Second order gauge transformation

In this appendix, we deal with first and second order gauge transformations. In order two obtain the second order waveform, it is necessary to derive the second order metric perturbations in an asymptotic flat (AF) gauge. The Regge-Wheeler-Zerilli formalism that we have employed in the RW gauge is not asymptotically flat. Therefore, we will focus on the gauge transformation from the RW gauge to an AF gauge. We also need to discuss the first order gauge transformation to an AF gauge simultaneously.

Here, we consider the following gauge transformation [35, 42].

$$x^\mu_{RW}\to x^\mu_{AF}=x^\mu_{RW}+\xi^{(1)\mu}\,(x^\alpha)+\frac{1}{2}\left[\xi^{(2)\mu}\,(x^\alpha)+\xi^{(1)\nu}\xi^{(1)\mu}{}_{,\nu}\,(x^\alpha)\right]\,, \tag{C.1}$$

where comma "," in the index indicates the partial derivative with respect to the background coordinates, and $\xi^{(1)\mu}$ and $\xi^{(2)\mu}$ are generators of the first and second order gauge transformations, respectively. Then, the metric perturbations changes as

$$h^{(1)}_{RW\mu\nu}\to h^{(1)}_{AF\mu\nu}=h^{(1)}_{RW\mu\nu}-\mathcal{L}_{\xi^{(1)}}g_{\mu\nu}\,, \tag{C.2}$$
$$h^{(2)}_{RW\mu\nu}\to h^{(2)}_{AF\mu\nu}=h^{(2)}_{RW\mu\nu}-\frac{1}{2}\mathcal{L}_{\xi^{(2)}}g_{\mu\nu}+\frac{1}{2}\mathcal{L}^2_{\xi^{(1)}}g_{\mu\nu}-\mathcal{L}_{\xi^{(1)}}h^{(1)}_{RW\mu\nu}\,. \tag{C.3}$$

Next, we discuss the $(\ell=2)\cdot(\ell=2)$ and $(\ell=0)\cdot(\ell=2)$ parts separately.

Appendix C.1. First order $\ell=2$ mode and second order $(\ell=2)\cdot(\ell=2)$ part

In this paper, we have used only the even parity mode, therefore a generator of the gauge transformation for $\ell=2$, $m=0$ modes can be written as

$$\xi^{(i)\mu}_{\ell=2}=\Big\{V_0^{(i)}(t,r)Y_{20}(\theta,\phi),\,V_1^{(i)}(t,r)Y_{20}(\theta,\phi),$$
$$V_2^{(i)}(t,r)\partial_\theta Y_{20}(\theta,\phi),\,V_2^{(i)}(t,r)\frac{\partial_\phi Y_{20}(\theta,\phi)}{\sin^2\theta}\Big\}\,, \tag{C.4}$$

where $i = 1$ and 2 denote the first and second perturbative order, respectively. There are three degrees of gauge freedom for each perturbative order.

The gauge transformation of the metric perturbations is explicitly given as follows: For the first order metric perturbations, we find

$$
\begin{aligned}
H_{0\,20}^{(1)AF}(t,r) &= H_{0\,20}^{(1)RW}(t,r) + 2\,\frac{\partial}{\partial t}{V_0}^{(1)}(t,r) + 2\,\frac{M}{r\,(r-2\,M)}{V_1}^{(1)}(t,r)\ ,\\
H_{1\,20}^{(1)AF}(t,r) &= H_{1\,20}^{(1)RW}(t,r) + \frac{(r-2\,M)}{r}\frac{\partial}{\partial r}{V_0}^{(1)}(t,r) - \frac{r}{r-2\,M}\frac{\partial}{\partial t}{V_1}^{(1)}(t,r)\ ,\\
H_{2\,20}^{(1)AF}(t,r) &= H_{2\,20}^{(1)RW}(t,r) - 2\,\frac{\partial}{\partial r}{V_1}^{(1)}(t,r) + 2\,\frac{M}{r\,(r-2\,M)}{V_1}^{(1)}(t,r)\ ,\\
K_{20}^{(1)AF}(t,r) &= K_{20}^{(1)RW}(t,r) - \frac{2}{r}{V_1}^{(1)}(t,r)\ ,\\
h_{0\,20}^{(e)(1)AF}(t,r) &= \frac{(r-2\,M)}{r}{V_0}^{(1)}(t,r) - r^2\frac{\partial}{\partial t}{V_2}^{(1)}(t,r)\ ,\\
h_{1\,20}^{(e)(1)AF}(t,r) &= -\frac{r}{r-2\,M}{V_1}^{(1)}(t,r) - r^2\frac{\partial}{\partial r}{V_2}^{(1)}(t,r)\ ,\\
G_{20}^{(1)AF}(t,r) &= -2\,{V_2}^{(1)}(t,r)\ .
\end{aligned}
\tag{C.5}
$$

For the second order metric perturbations, we can calculate the gauge transformation straightforwardly, but we obtain very long expressions. For example, they can be written formally as

$$
K_{20}^{(2)AF}(t,r) = K_{20}^{(2)RW}(t,r) - \frac{1}{r}{V_1}^{(2)}(t,r) + \delta K_{20}^{(2)}(t,r)\,,
\tag{C.6}
$$

$$
h_{1\,20}^{(e)(2)AF}(t,r) = -\frac{r}{2(r-2\,M)}{V_1}^{(2)}(t,r) - \frac{r^2}{2}\frac{\partial}{\partial r}{V_2}^{(2)}(t,r) + \delta h_{1\,20}^{(2)(e)}(t,r)\,.
\tag{C.7}
$$

$$
G_{20}^{(2)AF}(t,r) = -{V_2}^{(1)}(t,r) + \delta G_{20}^{(2)}(t,r)\,,
\tag{C.8}
$$

where $\delta K_{20}^{(2)}$, $\delta h_{1\,20}^{(2)(e)}$ and $\delta G_{20}^{(2)}$ are defined by the tensor harmonics expansion of the last two terms in the right hand side of (C.3). This includes only quadratic terms of the first order wave-function.

First, we consider the asymptotic behavior on the $\ell = 2$ mode of the first order metric perturbations in the RW gauge. The asymptotic expansion of the wave-function $\psi_{20}^{\rm even}$ is given by

$$
\begin{aligned}
\psi_{20}^{\rm even}(t,r) = {}& \frac{1}{3}\,\frac{d^2}{d{T_r}^2}F(T_r) + \left(\frac{d}{dT_r}F(T_r)\right)r^{-1} + \left(F(T_r) - M\frac{d}{dT_r}F(T_r)\right)r^{-2}\\
&+\mathrm{O}(r^{-3})\,,
\end{aligned}
\tag{C.9}
$$

where we have introduced $T_r = t - r_*(r)$ for simplicity. In the following calculation, we need only the leading order contribution with respect to the above large r expansion. Then, the coefficients of the metric perturbations are given, from (A.11), as follows

$$
\begin{aligned}
H_{0\,20}^{(1)RW}(t,r) = H_{2\,20}^{(1)RW}(t,r) &= \frac{1}{3}\,\left(\frac{d^4}{d{T_r}^4}F(T_r)\right)r + \mathrm{O}(r^0)\,,\\
H_{1\,20}^{(1)RW}(t,r) &= -\frac{1}{3}\,\left(\frac{d^4}{d{T_r}^4}F(T_r)\right)r + \mathrm{O}(r^0)\,,
\end{aligned}
$$

$$K_{20}^{(1)RW}(t,r) = -\frac{1}{3}\frac{d^3}{d{T_r}^3}F\left(T_r\right) + \mathrm{O}(r^{-1})\,. \tag{C.10}$$

On the other hand, the metric perturbations in an AF gauge should behave as

$$\begin{aligned}
&H_{0\,20}^{(1)AF}(t,r) = H_{1\,20}^{(1)AF}(t,r) = h_{0\,20}^{(e)(1)AF}(t,r) = 0\,. \quad H_{2\,20}^{(1)AF}(t,r) = \mathrm{O}(r^{-3})\,,\\
&h_{1\,20}^{(e)(1)AF}(t,r) = \mathrm{O}(r^{-1})\,, K_{20}^{(1)AF}(t,r) = \mathrm{O}(r^{-1})\,, \quad G_{20}^{(1)AF}(t,r) = \mathrm{O}(r^{-1})\,.
\end{aligned} \tag{C.11}$$

This asymptotic behavior will also be considered for the second order calculation. We find the following gauge transformation to go to the AF gauge.

$$\begin{aligned}
V_0^{(1)}(t,r) &= -\frac{1}{6}\left(\frac{d^3}{d{T_r}^3}F\left(T_r\right)\right) r + \mathrm{O}(r^0)\,,\\
V_1^{(1)}(t,r) &= -\frac{1}{6}\left(\frac{d^3}{d{T_r}^3}F\left(T_r\right)\right) r + \mathrm{O}(r^0)\,,\\
V_2^{(1)}(t,r) &= -\frac{1}{6}\left(\frac{d^2}{d{T_r}^2}F\left(T_r\right)\right) r^{-1} + \mathrm{O}(r^{-2})\,.
\end{aligned} \tag{C.12}$$

The above results are calculated iteratively for large r expansion. Since the transverse-traceless tensor harmonics for the even parity part is $\boldsymbol{f}_{\ell m}$ in (A.1), the coefficient of the metric perturbations related to the gravitational wave is $G_{\ell m}^{(1)AF}$. This becomes

$$\begin{aligned}
G_{20}^{(1)AF}(t,r) &= \frac{1}{3}\frac{1}{r}\frac{d^2}{d{T_r}^2}F\left(T_r\right) + \mathrm{O}(r^{-2})\\
&= \frac{1}{r}\psi_{20}^{\mathrm{even}}(t,r) + \mathrm{O}(r^{-2})\,.
\end{aligned} \tag{C.13}$$

with the use of (C.9)

Next, we discuss the second perturbative order. When we treat the second order metric perturbations from the $(\ell=2)\cdot(\ell=2)$ coupling, in practice, we calculate $\tilde{\chi}_{20}^{\mathrm{Z}}$ numerically instead of $\chi_{20}^{\mathrm{Z}}(t,r)$, where $\tilde{\chi}_{20}^{\mathrm{Z}}$ has been considered in (55) as

$$\tilde{\chi}_{20}^{\mathrm{Z}}(t,r) = \chi_{20}^{\mathrm{Z}}(t,r) - \chi_{20}^{\mathrm{reg},(2,2)}(t,r) - \chi_{20}^{\mathrm{reg},(0,2)}(t,r)\,, \tag{C.14}$$

where $\chi_{20}^{\mathrm{reg},(0,2)}$ is the $(\ell=0)\cdot(\ell=2)$ contribution, to be discussed in the next subsection. Here, to derive the gravitational wave amplitude for the second perturbative order, we also need to obtain the coefficient $G_{20}^{(2)AF}$ in an AF gauge as in the first order case.

The asymptotic expansion of $\tilde{\chi}_{20}^{\mathrm{Z}}$ is

$$\tilde{\chi}_{20}^{\mathrm{Z}}(t,r) = \frac{1}{3}\frac{d^3}{d{T_r}^3}F_2\left(T_r\right) + \mathrm{O}(r^{-1})\,. \tag{C.15}$$

Here, we may consider only the leading order contribution with respect to large r expansion in the same manner as the first order calculation. The $\tilde{\chi}_{20}^{\mathrm{Z}}$ contribution to the waveform is derived by the same method as that for the first perturbative order.

First, we obtain $\partial K_{20}^{(2)RW}/\partial t$ in the RW gauge from (51) as

$$\begin{aligned}
\frac{\partial}{\partial t}K_{20}^{(2)RW}(t,r) = &-\frac{1}{3}\frac{d^4}{d{T_r}^4}F_2\left(T_r\right) + \frac{\sqrt{5}}{18\sqrt{\pi}}\left(\frac{d^4}{d{T_r}^4}F\left(T_r\right)\right)\frac{d^3}{d{T_r}^3}F\left(T_r\right)\\
&+\mathrm{O}(r^{-1})\,,
\end{aligned} \tag{C.16}$$

where the second term in the right hand side of the above equation arises from the regularization function $\chi_{20}^{\mathrm{reg},(2,2)}$ and the $\mathcal{A}_{20}^{(1)}$ and $\mathcal{B}_{20}^{(0)}$-terms in (51). Integrating the above equation for $K_{20}^{(2)RW}$

$$K_{20}^{(2)RW}(t,r) = -\frac{1}{3}\frac{d^3}{d{T_r}^3}F_2(T_r) + \frac{\sqrt{5}}{36\sqrt{\pi}}\left(\frac{d^3}{d{T_r}^3}F(T_r)\right)^2 + \mathrm{O}(r^{-1}) \,. \tag{C.17}$$

The second order gauge transformation of $K_{20}^{(2)}$ is given by

$$K_{20}^{(2)AF}(t,r) = K_{20}^{(2)RW}(t,r) - \frac{1}{r}{V_1}^{(2)}(t,r) + \delta K_{20}^{(2)}(t,r) \,, \tag{C.18}$$

where, $\delta K_{20}^{(2)}$ is defined by the tensor harmonics expansion of $(1/2)\mathcal{L}_{\xi^{(1)}}^2 g_{\mu\nu} - \mathcal{L}_{\xi^{(1)}} h_{RW\mu\nu}^{(1)}$ in (C.3) and derived as

$$\delta K_{20}^{(2)}(t,r) = \frac{\sqrt{5}}{252\sqrt{\pi}}\left[\left(\frac{d^4}{d{T_r}^4}F(T_r)\right)\frac{d^2}{d{T_r}^2}F(T_r) - 2\left(\frac{d^3}{d{T_r}^3}F(T_r)\right)^2\right] \\ +\mathrm{O}(r^{-1}) \,. \tag{C.19}$$

From the above results and the AF gauge condition for $K_{20}^{(2)}$ in (C.11), in order to remove the $O(r^0)$ terms, the second order gauge transformation ${V_1}^{(2)}$ is

$${V_1}^{(2)}(t,r) = -\frac{1}{3}\left(\frac{d^3}{d{T_r}^3}F_2(T_r)\right) r \\ +\frac{\sqrt{5}}{252\sqrt{\pi}}\left[5\left(\frac{d^3}{d{T_r}^3}F(T_r)\right)^2 + \left(\frac{d^4}{d{T_r}^4}F(T_r)\right)\frac{d^2}{d{T_r}^2}F(T_r)\right] r + \mathrm{O}(r^0) \,. \tag{C.20}$$

Here, we note that it is sufficient to consider the leading order with respect to large r to derive the second order waveform.

Next, we consider to derive ${V_2}^{(2)}$ from the condition of $h_{1\,20}^{(e)(2)AF}$. The second order gauge transformation is given by

$$h_{1\,20}^{(e)(2)AF}(t,r) = -\frac{r}{2(r-2M)}{V_1}^{(2)}(t,r) - \frac{r^2}{2}\frac{\partial}{\partial r}{V_2}^{(2)}(t,r) + \delta h_{1\,20}^{(e)(2)}(t,r) \,. \tag{C.21}$$

$\delta h_{1\,20}^{(e)(2)}$ is defined by the tensor harmonics expansion of $(1/2)\mathcal{L}_{\xi^{(1)}}^2 g_{\mu\nu} - \mathcal{L}_{\xi^{(1)}} h_{RW\mu\nu}^{(1)}$ in Eq. (C.3). By considering the asymptotic expansion, we obtain

$$\delta h_{1\,20}^{(e)(2)}(t,r) = \frac{\sqrt{5}}{504\sqrt{\pi}}\left[\left(\frac{d^4}{d{T_r}^4}F(T_r)\right)\frac{d^2}{d{T_r}^2}F(T_r) + \left(\frac{d^3}{d{T_r}^3}F(T_r)\right)^2\right] r \\ +\mathrm{O}(r^0) \,. \tag{C.22}$$

Then, ${V_2}^{(2)}$ is calculated from the above value and the result for ${V_1}^{(2)}$ in (C.20) with the AF gauge condition in Eq.(C.11) as

$$\frac{\partial}{\partial r}{V_2}^{(2)}(t,r) = \frac{1}{3}\frac{d^3}{d{T_r}^3}F_2(T_r)\, r^{-1} - \frac{\sqrt{5}}{63\sqrt{\pi}}\left(\frac{d^3}{d{T_r}^3}F(T_r)\right)^2 r^{-1} + \mathrm{O}(r^{-2}) \\ = -\frac{\partial}{\partial t}{V_2}^{(2)}(t,r) + \mathrm{O}(r^{-2}) \,. \tag{C.23}$$

In the last line, we have used the definition of the retarded time, $T_r = t - r_*(r)$.

From the above results, we can consider the metric perturbations related to the gravitational wave amplitude, i.e., $G_{20}^{(2)AF}$. The gauge transformation is given by

$$G_{20}^{(2)AF}(t,r) = -{V_2}^{(1)}(t,r) + \delta G_{20}^{(2)}(t,r)\,. \tag{C.24}$$

Here, $\delta G_{20}^{(2)}$ is defined by the tensor harmonics expansion of $(1/2)\mathcal{L}^2_{\xi^{(1)}} g_{\mu\nu} - \mathcal{L}_{\xi^{(1)}} h^{(1)}_{RW\mu\nu}$ in Eq. (C.3), and found to be

$$\delta G_{20}^{(2)}(t,r) = \frac{\sqrt{5}}{126\sqrt{\pi}} \left(\frac{d^2}{d{T_r}^2} F(T_r)\right)\left(\frac{d^3}{d{T_r}^3} F(T_r)\right) r^{-1} + \mathrm{O}(r^{-2})\,. \tag{C.25}$$

Inserting (C.23) and (C.25) into (C.24), we obtain

$$\begin{aligned}\frac{\partial}{\partial t} G_{20}^{(2)}(t,r) &= \frac{1}{3}\frac{d^3}{d{T_r}^3} F_2(T_r)\, r^{-1} \\ &+ \frac{\sqrt{5}}{126\sqrt{\pi}}\left[-\left(\frac{d^3}{d{T_r}^3} F(T_r)\right)^2 + \left(\frac{d^4}{d{T_r}^4} F(T_r)\right)\frac{d^2}{d{T_r}^2} F(T_r)\right] r^{-1} + \mathrm{O}(r^{-2})\,. \end{aligned}\tag{C.26}$$

The gravitational waveform is now (by use of $\psi_{20}^{\rm even}$ and $\tilde{\chi}_{20}^{\rm Z}$)

$$\begin{aligned}\frac{\partial}{\partial t} G_{20}^{(2)}(t,r) &= \frac{1}{r}\tilde{\chi}_{20}^{\rm Z}(t,r) \\ &+ \frac{\sqrt{5}}{14\sqrt{\pi}}\frac{1}{r}\left[-\left(\frac{\partial}{\partial t}\psi_{20}^{\rm even}(t,r)\right)^2 + \psi_{20}^{\rm even}(t,r)\frac{\partial^2}{\partial t^2}\psi_{20}^{\rm even}(t,r)\right] + \mathrm{O}(r^{-2})\,. \end{aligned}\tag{C.27}$$

If we consider higher order corrections with respect to the $1/r$ expansion, we can show that all metric components satisfy the asymptotic flat gauge condition in (C.11).

Appendix C.2. Second order $(\ell = 0)\cdot(\ell = 2)$ part

We have already discussed the first order perturbations for the $\ell = 0$ mode in section 3.4. In this paper, we use the first order $\ell = 0$ metric perturbation given by (44). This satisfies the AF gauge condition in (C.11). Therefore, it is not necessary to consider the first order gauge transformation of the $\ell = 0$ mode in (C.2) and (C.3), and we will focus on the second perturbative order related to the $(\ell = 0)\cdot(\ell = 2)$ coupling.

We discuss the gravitational wave amplitude for the second perturbative order which arises from the $(\ell = 0)\cdot(\ell = 2)$ coupling by using the same method as in the case of the $(\ell = 2)\cdot(\ell = 2)$ part. Note that we have already discussed the contribution from $\tilde{\chi}_{20}^{\rm Z}$ and $\chi_{20}^{{\rm reg},(2,2)}$ in the second order wave function of (55). In the following, we consider the $\chi_{20}^{{\rm reg},(0,2)}$ and the contribution of the last term in the right hand side of (C.3), i.e., $-\mathcal{L}_{\xi^{(1)}} h^{(1)}_{RW\mu\nu}$ where $\xi^{(1)}$ is the generator of the gauge transformations for $\ell = 2$, and we use (44) as the $\ell = 0$ mode of $h^{(1)}_{RW\mu\nu}$.

The regularization function $\chi_{20}^{{\rm reg},(0,2)}$ is given in (62). This function behaves $\mathrm{O}(r^{-4})$ for large r, therefore we expect that $\chi_{20}^{{\rm reg},(0,2)}$ does not contribute to the second order gravitational waveform at infinity.

In the same way as for the $(\ell = 2)\cdot(\ell = 2)$ part, we consider the gauge transformation of $K_{20}^{(2)}$ in (C.18). Here, we obtain

$$K_{20}^{(2)RW}(t,r) + \delta K_{20}^{(2)}(t,r) = O(r^{-3})\,, \tag{C.28}$$

where $K_{20}^{(2)RW}$ and $\delta K_{20}^{(2)}$ arise from $\chi_{20}^{\mathrm{reg},(0,2)}$ and the tensor harmonics expansion of $-\mathcal{L}_{\xi^{(1)}} h_{RW\mu\nu}^{(1)}$ in (C.3), respectively. This is already asymptotically flat, therefore we do not need any further gauge transformation, ${V_1}^{(2)}$ for the $(\ell = 0) \cdot (\ell = 2)$ part.

$$ {V_1}^{(2)}(t,r) = \mathrm{O}(r^{-2}) \,. \tag{C.29} $$

In (C.21), we have

$$ \delta h_{1\,20}^{(2)(e)}(t,r) = \mathrm{O}(r^{-3}) \,, \tag{C.30} $$

where $\delta h_{1\,20}^{(2)(e)}$ is defined by the tensor harmonics expansion of $-\mathcal{L}_{\xi^{(1)}} h_{RW\mu\nu}^{(1)}$ in (C.3). From this equation, we also conclude

$$ \frac{\partial}{\partial r} {V_2}^{(2)}(t,r) = \mathrm{O}(r^{-4}) \,. \tag{C.31} $$

Hence, there is no contribution from the $(\ell = 0)\cdot(\ell = 2)$ coupling to the second order gravitational wave, except for $\tilde{\chi}_{20}^{\mathrm{Z}}$. This means that if we obtain $\tilde{\chi}_{20}^{\mathrm{Z}}$ in the numerical calculation, we can obtain the gravitational wave amplitude for the second perturbative order by using (C.27).

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